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SOME THEOREMS ON SPECIAL PROJECTIVE MOTION IN A SPECIAL SYMMETRIC FINSLER SPACE

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We consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\, dt \qquad \text{(A)}$$

where $v^i(x)$ is any vector field and dt an infinitesimal constant. We define that the necessary and sufficient condition that (A) be a special projective motion in F_n is that the Lie-derivative of $\pi^i_{jk}(x, \dot{x})$ with (A) itself has the form

$$\underset{v}{\mathscr{L}}\, \pi^i_{jk}(x, \dot{x}) = \delta^i_j \varepsilon_k + \delta^i_k \varepsilon_j + \phi_{jk} v^i ,$$

where ε_k is any non-zero covariant vector and

$$\phi_{jk} = \phi_{kj} .$$

Certain theorems concerning the special projective motion and special curvature collineations have been obtained in a Finsler space F_n.

1. Introduction

Let us consider an n-dimensional Finsler space F_n [1] in which the metric tensors $g_{ij}(x, \dot{x})$ and $g^{ij}(x, \dot{x})$ are symmetric in their indices i and j and are homogeneous of degree zero in $\dot{x}^i$.

The projective covariant derivative [3] of a vector field $X^i(x, \dot{x})$ is given by

$$X^i_{((k))} = \partial_k X^i - (\dot{\partial}_h X^i)\, \pi^h_{rk}\, \dot{x}^r + X^h\, \pi^i_{hk} \text{ [1]} \tag{1.1}$$

where

$$\pi^i_{hk}(x, \dot{x}) \overset{\text{def.}}{=\!=} \left\{ G^i_{hk} - \frac{1}{n+1}\,(2\,\delta^i_{(h}\, G^r_{k)r} + \dot{x}^i\, G^r_{rkh}) \right\} \tag{1.2}$$

is called projective connection coefficient and is also positively homogeneous function of degree zero in $\dot{x}^i$ and satisfies the following identities :

[1]) $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial \dot{x}^i$.

a) $\pi^i_{hkr}\,\dot{x}^r = 0$ b) $\dot{\partial}_j\,\pi^i_{hk} = \pi^i_{jhk}$ and (1.3)

c) $\pi^i_{hk}\,\dot{x}^h = \pi^i_k$;

we have the following commutation formulae :

$$\dot{\partial}_h\,(T^i_{j((m))}) - \dot{\partial}_h\,T^i_{j((m))} = T^r_j\,\pi^i_{rhm} - T^i_r\,\pi^r_{jhm} \tag{1.4}$$

and

$$2T^i_{j[((h))\,((k))]} = -\,(\dot{\partial}_r\,T^i_j)\,Q^r_{hk} + T^r_j\,Q^i_{rhk} - T^i_r\,Q^r_{jhk}\,, \tag{1.5}$$

where

$$Q^i_{jkh}\,(x,\dot{x}) \overset{\text{def.}}{=} 2\,\{\partial_{]h}\pi^i_{k]j} - (\dot{\partial}_r\,\pi^i_{j[k})\,\pi^r_{h]s}\,\dot{x}^s + \pi^r_{j\,[k}\pi^i_{h]r}\}\,^{2)} \tag{1.6}$$

are called projective entities and satisfy the following identities [3] :

$$\text{a)}\quad Q^i_{hji} = Q_{hj}\,,\quad \text{b)}\quad Q^i_{hk}\,\dot{x}^h = Q^i_k\,,\quad \text{c)}\quad Q^i_{jhk}\,\dot{x}^j = Q^i_{hk} \tag{1.7}$$

and

d) $Q^i_{ki} = Q_k$.

Let us consider an infinitesimal point transformation :

$$\dot{x}^i = x^i + v^i\,(x)\,dt\,, \tag{1.8}$$

where $v^i\,(x)$ is any vector field and dt an infinitesimal constant. The Lie-derivative of a tensor field T^i_j and the projective connection coefficient is given by

$$\underset{v}{\mathscr{L}}\,T^i_j = T^i_{j((h))}\,V^h - T^h_j\,V^i_{((h))} + T^i_h\,V^h_{((j))} + (\dot{\partial}_h\,T^i_j)\,V^h_{((r))}\dot{x}^r \tag{1.9}$$

and

$$\underset{v}{\mathscr{L}}\,\pi^i_{jk} = V^i_{((j))\,((k))} + Q^i_{hjk}\,V^h + \pi^i_{jkh}\,\dot{x}^r v^h_{((r))}\,. \tag{i.i0}$$

We have the following commutation formulae :

$$\dot{\partial}_k\,(\underset{v}{\mathscr{L}}\,T^i_j) - \underset{v}{\mathscr{L}}\,(\dot{\partial}_k\,T^i_j) = 0 \tag{i.11}$$

$$\underset{v}{\mathscr{L}}\,T^i_{j((k))} - (\underset{v}{\mathscr{L}}\,T^i_j)_{((k))} = T^h_j\,\underset{v}{\mathscr{L}}\,\pi^i_{kh} - T^i_h\,\underset{v}{\mathscr{L}}\,\pi^h_{kj} - \tag{1.12}$$

$$-\,(\dot{\partial}_h\,T^i_j)\,\underset{v}{\mathscr{L}}\,\pi^h_{ks}\,\dot{x}^s$$

and

$$2\,\underset{v}{\mathscr{L}}\,\pi^i_{h\,[k((j))]} = \underset{v}{\mathscr{L}}\,Q^i_{jkh} + 2\,\underset{v}{\mathscr{L}}\,\pi^s_{m[j}\pi^i_{k]sh}\,\dot{x}^m\,. \tag{1.13}$$

We have the following definitions :

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2) $2A_{(hk)} = A_{hk} + A_{kh}$ and $2A_{[hk]} = A_{hk} - A_{kh}$.

Special projective curvature collineations [4]: An F_n is said to admit a special projective curvature collineation if there exists a vector V^i such that

$$\mathop{\mathscr{L}}_{v} Q^i_{jkh} = 0 . \tag{1.14}$$

Special projective Ricci collineations [4]: An F_n is said to admit a special projective Ricci collineation provided there exists a vector V^i such that

$$\mathop{\mathscr{L}}_{v} Q_{jk} = 0 . \tag{1.15}$$

Definition 1.1. A finsler space F_n is said to be a special projective symmetric space if the projective covariant derivative of $Q^i_{hjk}(x,\dot{x})$ satisfies the relation

$$Q^i_{hjk((r))} = 0 . \tag{1.16}$$

In a special projective symmetric Finsler space we have the following identities:

$$\text{a)}\quad Q^i_{jk((r))} = 0 , \qquad \text{b)}\quad Q^i_{k((r))} = 0 , \qquad \text{c)}\quad Q_{((r))} = 0. \tag{1.17}$$

2. Special curvature collineation and special projective motion

Definition 2.1. The necessary and sufficient condition that (1.8) be a special projective motion in F_n is that the Lie-derivative of $\pi^i_{jk}(x, \dot{x})$ with (1.8) itself has the form [5]:

$$\mathop{\mathscr{L}}_{v} \pi^i_{jk}(x, \dot{x}) = \delta^i_j\, \varepsilon_k + \delta^i_k\, \varepsilon_j + \phi_{jk}\, V^i , \tag{2.1}$$

where $\varepsilon_k(x)$ is any non-zero covariant vector and

$$\phi_{jk} = \phi_{kj} . \tag{2.2}$$

In view of (2.1), (1.3) b) we have

$$\mathop{\mathscr{L}}_{v} \pi^i_{hjk} = \phi_{hjk} V^i , \tag{2.3}$$

where

$$\phi_{hjk} \overset{\text{def.}}{=} \dot{\partial}_h\, \phi_{jk} . \tag{2.4}$$

In view of (1.13) and (2.1) the Lie-derivative of $Q^i_{jkh}(x, \dot{x})$ takes the following form :

$$\begin{aligned} \mathop{\mathscr{L}}_{v} Q^i_{jkh} = {} & \delta^i_k\, \varepsilon_{h((j))} - \delta^i_j\, \varepsilon_{h((k))} + (\phi_{hk((j))} - \phi_{hj((k))})\, v^i + \\ & + \phi_{hk}\, V^i_{((j))} - \phi_{hj}\, V^i_{((k))} + \\ & + (\phi_{mk}\, \pi^i_{jsh} - \phi_{mj}\, \pi^i_{ksh})\, V^s \dot{x}^m , \end{aligned} \tag{2.5}$$

where we have used

$$\text{a)}\quad \varepsilon_1\, \dot{x}^1 = 0 \quad \text{and} \quad \text{b)}\quad \varepsilon_{j((k))} = \varepsilon_{k((j))} . \tag{2.6}$$

If a special projective motion of F_n becomes a special projective curvature collineation the formula (2.5) takes the form

$$\delta^i_k \varepsilon_{h((j))} - \delta^i_j \varepsilon_{h((k))} + (\phi_{hk((j))} - \phi_{hj((k))}) V^i + \phi_{hk} V^i_{((j))} - \phi_{hj} V^i_{((k))} + (\phi_{mk} \pi^i_{jsh} - \phi_{mj} \pi^i_{ksh}) V^s \dot{x}^m = 0 . \tag{2.7}$$

Transvecting (2.7) by $\dot{x}^h$ and noting (1.3) a), (2.6) we obtain

$$\{ (\phi_{hk((j))} - \phi_{hj((k))}) v^i + \phi_{hk} V^i_{((j))} - \phi_{hj} V^i_{((k))} \} \dot{x}^h = 0 , \tag{2.8}$$

where we have used the fact that $x^i_{((k))} = 0$.

Thus we have :

Theorem 2.1. In a Finsler space F_n if a special projective motion characterized by (2.1) becomes a special projective curvature collineation, then equation (2.8) holds.

Contracting (2.5) w.r.t. indices i and h and using (1.7) a) we get

$$\underset{v}{\mathscr{L}} Q_{jk} = \delta^i_k \varepsilon_{i((j))} - \delta^i_j \varepsilon_{i((k))} + (\phi_{ik((j))} - \phi_{ij((k))}) V^i + \phi_{ik} V^i_{((j))} - \phi_{ij} V^i_{((k))} + (\phi_{mk} \pi^i_{jsi} - \phi_{mj} \pi^i_{ksi}) V^s \dot{x}^m . \tag{2.9}$$

If F_n admits a special projective Ricci collineation, then in view of (1.15), (2.9) takes the form

$$\delta^k_i \varepsilon_{i((j))} - \delta^i_j \varepsilon_{i((k))} + (\phi_{ik((j))} - \phi_{ij((k))}) V^i + \phi_{ik} V^i_{((j))} - \phi_{ij} V^i_{((k))} + (\phi_{mk} \pi^i_{jsi} - \phi_{mj} \pi^i_{ksi}) V^s \dot{x}^m = 0 . \tag{2.10}$$

Transvecting (2.10) by $\dot{x}^j$ and noting (1.3) a), we get

$$\{ (\phi_{ik((j))} - \phi_{ij((k))}) V^i + \phi_{ik} V^i_{((j))} - \phi_{ij} V^i_{((k))} - \phi_{mj} \pi^i_{ksi} V^s \dot{x}^m \} \dot{x}^j = 0 . \tag{2.11}$$

Thus we have :

Theorem 2.2. In an F_n if a special projective motion becomes a special projective Ricci collineation, then the result (2.11) holds.

3. Special projective symmetric space

Applying the commutation formula (1.12) to the projective entity $Q^i_j(x,\dot{x})$, we get

$$\underset{v}{\mathscr{L}} Q^i_{j((k))} - (\underset{v}{\mathscr{L}} Q^i_j)_{((k))} = Q^h_j (\underset{v}{\mathscr{L}} \pi^i_{kh}) - Q^i_h (\underset{v}{\mathscr{L}} \pi^h_{kj}) - (\dot{\partial}_h Q^i_j)(\underset{v}{\mathscr{L}} \pi^h_{ks} \dot{x}^s) . \tag{3.1}$$

In view of (1.7) a), (1.17) b), (2.1), (2.6), the equation (3.1) takes the form :

$$(\underset{v}{\mathscr{L}} Q^i_j)_{((k))} = Q^i_k \varepsilon_j + Q^i_k \phi_{kj} V^h - Q^h_j \delta^i_k \varepsilon_h - Q^h_j \phi_{kh} V^i + \tag{3.2}$$
$$+ (\dot\partial_h Q^i_j) \phi_{hs} V^h \dot x^s .$$

Contracting (3.2) w.r.t. indices i and j we get

$$(\underset{v}{\mathscr{L}} Q^i_k)_{((k))} = Q^i_k \varepsilon_i + Q^i_h \phi_{ki} V^h - Q^h_k \varepsilon_h - Q^h_i \phi_{kh} V^i + \tag{3.3}$$
$$+ (\dot\partial_h Q^i_i) \phi_{kh} V^h \dot x^s .$$

Thus we have :

Theorem 3.1. In a special projective symmetric Finsler space if an infinitesimal transformation (1.8) defines a special projective motion, then equation (3.3) holds.

Transvecting (2.5) w.r.t. $\dot x^j$ and noting (1.7) c) we get

$$\underset{v}{\mathscr{L}} Q^i_{kh} = \delta^i_k \varepsilon_{h((j))} \dot x^j - \varepsilon_{h((k))} \dot x^i + \{ (\phi_{hk((j))} - \phi_{hj((k))}) V^i + \tag{3.4}$$
$$+ \phi_{hk} V^i_{((j))} - \phi_{hj} V^i_{((k))} - \phi_{mj} \pi^i_{ksh} V^s \dot x^m \} \dot x^j .$$

Applying the commutative formula (1.4) to the projective entity $Q^i_{jk}(x, \dot x)$ we obtain

$$\dot\partial_h (Q^i_{jk((m))}) - (\dot\partial_h Q^i_{jk})_{((m))} = Q^s_{jk} \pi^i_{hms} - Q^i_{sk} \pi^s_{jhm} + Q^i_{sj} \pi^s_{khm} \tag{3.5}$$

which in view of (1.16), (1.17) a) reduces to the form

$$Q^s_{jk} \pi^i_{hms} - Q^i_{sk} \pi^s_{jhm} + Q^i_{sj} \pi^s_{khm} = 0 . \tag{3.6}$$

Applying $\underset{v}{\mathscr{L}}$ operators to (3.6) and using (3.4), (2.3), we get

$$[\delta^s_j \varepsilon_{k((r))} \dot x^r - \varepsilon_{k((j))} \dot x^s + \{ (\phi_{kj((r))} - \phi_{kr((j))}) V^s + \tag{3.7}$$
$$+ \phi_{kj} V^s_{((r))} - \phi_{kr} V^s_{((j))} - \phi_{mr} \pi^s_{klh} V^l \dot x^m \} \dot x^r] \pi^i_{hms} -$$
$$- [\delta^i_s \varepsilon_{k((r))} \dot x^r - \varepsilon_{k((s))} \dot x^i + \{ (\phi_{ks((r))} - \phi_{kr((s))}) V^l +$$
$$+ \phi_{ks} V^i_{((r))} - \phi_{kr} V^i_{((s))} - \phi_{mr} \pi^i_{slk} V^l \dot x^m \} \dot x^r] \pi^s_{jhm} +$$
$$+ [\delta^i_s \varepsilon_{j((r))} \dot x^r - \varepsilon_{j((s))} \dot x^i + \{ (\phi_{js((r))} - \phi_{jr((s))}) V^i +$$
$$+ \phi_{js} V^i_{((r))} - \phi_{jr} V^i_{((s))} - \phi_{mr} \pi^i_{slj} V^l \dot x^m \} \dot x^r] \pi^s_{khm} +$$
$$+ Q^s_{jk} \phi_{hms} V^i - Q^i_{sk} \phi_{jhm} V^s + Q^i_{sj} \phi_{khm} V^s = 0 .$$

Transvecting (3.7) by $\dot x^h$ and using (1.3) a) we obtain

$$Q^s_{jk} \phi_{hms} V^i + (Q^i_{sj} \phi_{khm} - Q^i_{sk} \phi_{jhm}) V^s = 0 . \tag{3.8}$$

Thus we have :

Theorem 3.2. If a special projective symmetric Finsler space admits a special projective motion characterized by infinitesimal transformation (1.8), then equation (3.8) holds.

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ÖZET

Bu çalışmada, $v^i(x)$ herhangi bir vektör alam ve dt bir infinitezimal sabit olmak üzere,

$$\bar{x}^i = x^i + v^i(x)\, dt$$

infinitezimal nokta transformasyonu gözönüne alınmakta, bir F_n Finsler uzayında özel eğrisel kolineasyonlar ve özel projektif hareketle ilgili bir takım teoremler elde edilmektedir.