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HARMONIC FUNCTIONS ON FINSLER SPACES

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Let $(M^n(c), E)$ be a Finsler space of scalar curvature $c \neq 0$ and vanishing mixed torsion vector $P_j = \dot{\partial}_i N^i_j - F^i_{ji}$. All h-harmonic functions $f(x, y)$ on $T(M^n(c))$-{0} which are positive homogeneous of degree r in the y^i's and whose h-gradient has compact support, are given by $f = a\ E^{r/2}$; $a \in IR$. The image of a totally-geodesic immersion of a Finsler space in a Landsberg space M^{n+p} is not contained in any h-convex supporting set of M^{n+p}.

1. INTRODUCTION

The induced bundle $\pi^{-1}\, T(M) \longrightarrow V(M)$ of a Finsler space M carries a naturally defined Riemann bundle metric g_{ij}; its Sasaki lift G_{AB} makes $V(M)$ into a Riemann space and the horizontal distribution N (of the unique regular Cartan connection) appears to be precisely the (G_{AB})-orthogonal complement of the vertical distribution on $V(M)$. Moreover $V(M)$ has a natural orientation arising from the almost complex structure associated with N, cf. ref. [3]. The choice of $V(M) = T(M) - \{0\}$ (rather than the whole of $T(M)$) is prompted by the lack of differentiability of the Finsler energy function (only C^1 on $T(M)$) along the zero section (consequently g_{ij} are discontinuous at $y^i = 0$).

The study of the geometry of $(V(M), G_{AB})$ based on the Riemannian machinery has a highly complicate character, see [32]. In turn, in the framework of Finsler geometry, the presence of the (generally non-holonomic) Pfaffian system N on $V(M)$ yields decompositions of tensor fields on $V(M)$ in horizontal, vertical and mixed components (with respect to $T(V(M)) = N \oplus \text{Ker}\,(d\pi)$). For instance, the curvature tensors R^i_{jkm}, P^i_{jkm} and S^i_{jkm} occuring in E.CARTAN's theories (cf. [8]) are nothing but the horizontal, mixed and vertical parts of a unique $\pi^{-1}\, T(M)$ - valued curvature form $\widetilde{R}$ (of the Cartan connection ∇ in $\pi^{-1}\, T(M)$); these are usually handled independently, by analogy with their Riemannian counterparts.

In the present note we apply E.Cartan's ideas (cf. also [1]) and decompose the Laplace-Beltrami operator (on $V(M)$) associated with the Sasaki metric. This procedure gives rise to two differential operators Δ^h and Δ^v. We prove an analogue of the classical E.Hopf's lemma for the operator Δ^h. Precisely, we determine all positively homogeneous (of degree r) differentiable functions on $V(M)$ which satisfy $\Delta^h f \geqq 0$ everywhere, and whose h-gradient has compact support in $V(M)$, provided that M is a Finsler space of non-zero scalar curvature (in the sense of [6]) having a vanishing Vrànceanu vector P_j. These have the form $f = a L^r$, $a \in IR$, where L is the fundamental Lagrangian function of the Finsler space, and $\Delta^h f = 0$ (which agrees with $L_{|i} = 0$, cf. ref. [21, p.115]). In particular, Finsler spaces of non-zero scalar curvature do not admit h-harmonic (i.e. $\Delta^h f = 0$) positive homogeneous functions of degree zero and with supp (grad$^h f$) compact (other than the constant functions), provided $P_j = 0$. This is based on a theorem of [23], where the meaning of the equations $\frac{\delta f}{\delta x^i} = 0$, $1 \leqq i \leqq n$, on a Finsler space is explained.

For a given transformation ϕ of M we show that Δ^h is invariant under $d\phi$ if and only if ϕ is an isometry of the Finsler space.

In § 5, as an application of the notions in § 1 - § 3, we consider totally-geodesic submanifolds M^n of a Landsberg space M^{n+p}. Then M^n is a Landsberg space (with the induced Finsler structure) and has a vanishing (horizontal) second fundamental form H^i_{ab}. This is analogous to harmonicity (of the given immersion $f: M^n \longrightarrow M^{n+p}$) in Riemannian geometry, cf. [17]. We prove that $f(M^n)$ cannot be contained in a h-convex supporting subset of M^{n+p}, provided that M^n is totally-geodesic. For the theory of submanifolds in Finsler spaces see [26], [27], [2], [16].

2. THE LAPLACE-BELTRAMI OPERATOR OF THE SASAKI METRIC

Let (M, E) be an n-dimensional Finsler space; here $E: T(M) \longrightarrow IR$ denotes the *Finsler energy*, cf. [18], ch. II, $E = L^2$. Let (U, x^i) be a local coordinate system and $(\pi^{-1}(U), x^i, y^i)$ the induced coordinates on $V(M) = T(M) - \{0\}$, where $\pi: V(M) \longrightarrow M$ denotes the natural projection. Let $g_{ij} = \frac{1}{2} \partial^2 E / \partial y^i \partial y^j$ be the associated Finsler metric $(0, 2)$-tensor field. Let N be the distribution on $V(M)$ given by the Pfaffian system $dy^i + N^i_j dx^j = 0$, where N^i_j are given by (18.15) in [21, p.118]. Let G_{AB} be the *Sasaki lift* of g_{ij} to $V(M)$, cf. [31, p.111]. We may use the non-holonomic frame $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ on $V(M)$ such as to express the Laplace-Beltrami operator Δ of the Riemannian manifold $(V(M), G_{AB})$ (on functions); one obtains :

$$\Delta f = \frac{1}{g} \frac{\delta}{\delta x^i}\left(g\, g^{ji} \cdot \frac{\delta f}{\delta x^j}\right) + \frac{1}{g} \frac{\partial}{\partial y^i}\left(g\, g^{ji} \frac{\partial f}{\partial y^j}\right) - g^{kj} G^i_{jk} \frac{\delta f}{\delta x^i} \tag{2.1}$$

where G^i_{jk} are the coefficients of the *Berwald connection*, cf. (18.14) in [21, p.118]. Also we use the notations $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$, $g = \det(g_{ij})$. Here $f \in C^\infty(V(M))$. Our (2.1) suggests the following definitions:

$$\begin{aligned} \Delta^h f &= \frac{1}{\sqrt{g}} \frac{\delta}{\delta x^i}\left(\sqrt{g}\, g^{ji} \frac{\delta f}{\delta x^j}\right) \\ \Delta^v f &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^i}\left(\sqrt{g}\, g^{ji} \frac{\partial f}{\partial y^j}\right). \end{aligned} \tag{2.2}$$

It is easily seen that the definitions above do not depend upon the choice of local coordinates, such that (2.2) gives two globally defined differential operators on $V(M)$. We also set $\operatorname{grad}^h f = g^{ij} \frac{\delta f}{\delta x^i} X_j$, where $X_j(u) = \left(u, \frac{\partial}{\partial x^j}\bigg|_{\pi(u)}\right)$, for any $u \in \pi^{-1}(U)$; the definition of $\operatorname{grad}^h f$ does not depend upon the choice of local coordinates, and $\operatorname{grad}^h f$ is refered to as the *h-gradient* of f

Let $\pi^{-1} T(M) \longrightarrow V(M)$ be the pullback bundle of $T(M)$ by π. Note that grad^h is a $\pi^{-1} T(M)$-valued $I\!R$-linear mapping on $C^\infty(V(M))$. If X is a cross-section in $\pi^{-1}T(M)$ (i.e. a *Finsler vector field* on M), then locally $X = X^i(x, y) X_i$. We set $\operatorname{div}^h X = \frac{1}{\sqrt{g}} \frac{\delta}{\delta x^i} (\sqrt{g}\, X^i)$. The definition of $\operatorname{div}^h X$ is independent of the choice of local coordinates on M; note that $\Delta^h f = \operatorname{div}^h (\operatorname{grad}^h f)$. A function $f \in C^\infty(V(M))$ is said to be *h-harmonic* if $\Delta^h f = 0$.

Let d^h be the *exterior h-differentiation operator* on Finsler forms, c.f. [30], and also [5]. If $f \in C^\infty(V(M))$, then $d^h f = \frac{\delta f}{\delta x^i} \bar{d}x^i$, where $\bar{d}x^i|_u = (u, dx^i|_{\pi(u)})$, for any $u \in \pi^{-1}(U)$. It is known that d^h satisfies the complex condition $(d^h)^2 = 0$ iff $R^i_{jk} = 0$, where R^i_{jk} is given by (17.8) in [21, p.112].

Note that $\pi^{-1} T(M)$ becomes a Riemannian bundle in a natural manner, cf. also [10]. Let g denote its bundle metric. This extends in a natural manner to $\pi^{-1} T^*(M) \longrightarrow V(M)$, where $T^*(M)$ denotes the cotangent bundle over M, and the following formulae hold:

$$\begin{aligned} \operatorname{div}^h (f\bar{X}) &= f \operatorname{div}^h \bar{X} + (\beta \bar{X}) f \\ \Delta^h (f^2) &= 2 f \Delta^h f + 2\, g(d^h f, d^h f) \end{aligned} \tag{2.3}$$

for any $f \in C^\infty(V(M))$ and any Finsler vector field $\bar{X}$ on M. Here β: $\pi^{-1}T(M) \longrightarrow N$ denotes the *horizontal lift*, i.e. $\beta X_i = \frac{\delta}{\delta x^i}$.

3. A DIVERGENCE FORMULA

Let $X = (X^i, \dot{X}^i)$, $\dot{X}^i = -N^i_j X^j$ be a *horizontal tangent vector* field on $V(M)$. Its divergence (with respect to the Sasaki metric) is given by :

$$\operatorname{div} X = \operatorname{div}^h \overline{X} + X(\log\sqrt{g}) - G^i_{ij} X^j \tag{3.1}$$

where $\overline{X} = L X$. Here $L : T(V(M)) \longrightarrow \pi^{-1} T(M)$ denotes the bundle morphism given by $L \dfrac{\partial}{\partial x^i} = X_i$, $L \dfrac{\partial}{\partial y^i} = 0$. Let P^i_{jk} be the *mixed* P^1*-torsion* of the Cartan connection; consider also $P_j = P^i_{ij}$. This is refered to as the *Vrànceanu vector* associated with the P^1-torsion. See [33]. Using the formulae (1.12) - (1.14) in [25, p.235], one obtains :

$$G^i_{ij} = P_j + \frac{\delta}{\delta x^j}(\text{iog}\sqrt{g}). \tag{3.2}$$

Using (3.2) to substitute in (2.1), (3.1), we obtain :

$$\begin{aligned} \Delta f &= \Delta^h f + \Delta^v f + g(d^v f, d^v \log\sqrt{g}) - g(d^h f, P) \\ \operatorname{div}(\beta \overline{X}) &= \operatorname{div}^h \overline{X} - P\overline{X}. \end{aligned} \tag{3.3}$$

Here $d^v f = \dfrac{\partial f}{\partial y^i} \overline{d} x^i$, $P\overline{X} = P_i X^i$. By Green's theorem, e.g. [20, p.281], vol. I, one has :

$$\int_{V(M)} (\operatorname{div} X) * 1 = 0,$$

provided that X has compact support. Here $*1$ denotes the canonical Rieman nian measure associated with the Sasaki metric. Also $V(M)$ is orientable in a natural manner due to the presence of the almost complex structure $J\beta\overline{X} = \gamma\overline{X}$, $J\gamma\overline{X} = -\beta\overline{X}$, cf. [4]. Here $\gamma : \pi^{-1} T(M) \longrightarrow \operatorname{Ker}(d\pi)$ denotes the *vertical lift*, i.e. $\gamma X_i = \dfrac{\partial}{\partial y^i}$. Let $f \in C^\infty(V(M))$ such that $\overline{X} = \operatorname{grad}^h f$ has compact support. Then (3.3) leads to :

$$\int_{V(M)} (\operatorname{div}^h \overline{X}) *1 = \int_{V(M)} (P\overline{X}) *1. \tag{3.4}$$

Suppose from now on that (M, E) obeys $P = 0$. Let us assume that $\Delta^h f \geq 0$. Then by (3.4) the function f is h-harmonic. Also

$$0 = \int_{V(M)} \Delta^h (f^2/2) *1 = \int_{V(M)} f\,\Delta^h f *1 + \int_{V(M)} \|d^h f\|^2 *1$$

and consequently $\frac{\delta f}{\delta x^i}=0$, $1 \leqq i \leqq n$. Let us assume now that (M, E) is a Finsler space of scalar curvature c, i.e. $R^i_{jk} = h^i_k c_j - h^i_j c_k$, where $c_j = \frac{1}{3} E \frac{\partial c}{\partial y^j} + KL\, l_j$, $L\, l_j = \frac{1}{2} \frac{\partial E}{\partial y^j}$, $h^i_j = g^{ik} h_{kj}$, $h_{ij} = g_{ij} - l_i l_j$, cf. [21, p.168]. Suppose also that f is positive homogeneous of degree r in the y^i's and $c \neq 0$. By a result of [23], since f is h-covariaut constant, we obtain $f = a\, L^r$, for some real constant a. We have obtained the following generalization of the theorem of E.Hopf (cf. e.g. ref. [20, p. 338], vol. II) :

Theorem A. *Let (M, E) be a Finsler space of scalar curvature c, $c \neq 0$, having a vanishing Vrànceanu vector P and $f \in C^\infty(V(M))$ such that $\Delta^h f \geqq 0$ and $\overline{X} = \text{grad}^h f$ has compact support in $V(M)$. If f is positive homogeneous of degree r in the directional arguments then $f = a\, E^{r/2}$, $a \in IR$ (and $\Delta^h f = 0$). In particular, a Finsler space of non-zero scalar curvature obeying $P = 0$ has no h-harmonic positive homogeneous function of degree zero, except for constant functions.*

Remarks. i) Our theorem A might be completed for the case $c = 0$ as follows: if $c = 0$ then $R^i_{jk} = 0$ and the Pfaffian system $d y^i + N^i_j\, d x^j = 0$ is integrable. Then $f_{|i} = 0$ implies that f is constant on each maximal integral manifold of the non-linear connection N of the Cartan connection, see ref. [23, p.555]. Here $f_{|i} = \frac{\delta f}{\delta x^i}$.

ii) Note that the assumption on the Vrànceanu vector in theorem A may be relaxed to $\int_{V(M)} (P\, \overline{X}) *1 = 0$ for any gradient Finsler vector field $\overline{X}$ on M.

iii) A *Landsberg space* is a Finsler space whose Berwald connection $(N^i_j, G^i_{jk}, 0)$ is h-metrical, cf. [21, p.162]. If (M, E) is Landsberg, then $P^i_{jk} = 0$, by a result in [21]. Therefore the hypothesis $P_j = 0$ in our theorem A is verified (the converse is not true in general). Let $M^n(c)$, $c \neq 0$, be a Landsberg space of scalar curvature c. We distinguish two cases; if $n > 2$ then by the theorem of S. NUMATA [8], $M^n(c)$ is a Riemannian manifold of constant sectional curvature, and if this is the case on one hand theorem A contains the classical Hopf lemma (when f does not depend upon the directional arguments), and on the other one obtains the following :

Corollary. *Let $(M^n(c), g_{ij}(x))$ be a real space-form, $n>2$, $c \neq 0$. Any positive homogeneous (in the y^i's, of degree r) h-harmonic function $f \in C^\infty(V(M^n(c)))$ has* *the form* $f(x, y) = a\,(g_{ij}(x)\, y^i y^j)^{r/2}$, *provided* supp (grad^h f) *is compact, where* $a \in IR$.

Finally, if $n = 2$ then S.Numata's theorem does not apply, and our result is completely new.

iv) If (M, E) is a Riemannian manifold, then by (3.3) the Laplacian of the Sasaki lift of $g_{ij}(x)$ to $V(M)$ (or $T(M)$) is given by $\mathbf{A} f = \Delta^h f + \Delta^v f$, $f \in C^\infty(V(M))$. Clearly $\Delta^h(f^v) = \Delta_M f$, for any $f \in C^\infty(M)$, where $f^v = f \circ \pi$ is the *vertical lift* of f, while Δ_M denotes the Laplacian of $(M, g_{ij}(x))$.

4. ISOMETRIES OF FINSLER SPACES

Let (M, E) be a Finsler space and $\phi : M \longrightarrow M$ a transformation of M. It is said to be an *isometry* of (M, E) if:

$$E \circ (d\phi) = E\,, \tag{4.1}$$

cf. also [15]. Let $p \in M$ and (U, x^i), (V, x'^i) coordinate neighborhoods at p and $\phi(p)$, respectively. Then (4.1) might be written $E(x^i, y^j) = E\left(\phi^i(x), \dfrac{\partial \phi^i}{\partial x^j}(x)\, y^j\right)$ and consequently ϕ is an isometry iff:

$$g_{ij}(x, y) = g_{ij}(x', y')\,,\ y'^i = \frac{\partial \phi^i}{\partial x^j}(x)\, y^j. \tag{4.2}$$

Conversely, (4.2) yields (4.1), by the classical Euler theorem on positive homogeneous functions.

Let $f \in C^\infty(V(M))$ and A a linear transformation of $C^\infty(V(M))$ into itself; for a given diffeomorphism $\Psi : V(M) \longrightarrow V(M)$ we denote by A^Ψ the mapping $f \longrightarrow (A f^{\Psi^{-1}})^\Psi$, where $f^\Psi = f \circ \Psi^{-1}$. Then A is *invariant* by Ψ if $A^\Psi = A$. Let ϕ be an isometry of M; then $(\mathbf{A}^h)^{d\phi} = \Delta^h$. The converse is also true. The argumentation follows the steps in [19, p.388], such that the details might be left as an exercise to the reader. One obtains:

Theorem B. *Let* ϕ *be a transformation of* M. *Then* $d\,\phi$ *leaves* Δ^h *invariant if an isometry of the Finsler space* M.

5. SUBMANIFOLDS OF LANDSBERG SPACES

Let M^{n+p} be an $(n + p)$-dimensional Landsberg space and f an immersion of an n-dimensional manifold M^n in M^{n+p}. Let $f : x^i = x^i(u^1, \ldots, u^n)$ be the equations of M^n in M^{n+p}. We set $B^i_a(u) = \dfrac{\partial x^i}{\partial u^a}(u)$, rank $(B^i_a) = n$. Let $(N^i_j, F^i_{jk}, C^i_{jk})$ be the Cartan connection of M^{n+p} and $(N^a_b, F^a_{bc}, C^a_{bc})$ the *induced connection* on the submanifold. We set:

$$B^i_{ab} = \frac{\partial^2 x^i}{\partial u^a \partial u^b}, \quad B^{ij}_{ab} = B^i_a B^j_b .$$

We recall (cf. e.g. [16]) the *(horizontal) Gauss equation* of M^n in M^{n+p}, i.e.

$$B^i_{ab} + B^{jk}_{ab} F^i_{jk} + H^j_{a0} B^k_b C^i_{jk} = F^c_{ab} B^i_c + H^i_{ab} . \tag{5.1}$$

Here H^i_{ab} denotes the *(horizontal) second fundamental form* of f, while $H^i_{a0} = H^i_{ab} v^b$. Here (u^a, v^a) are the naturally induced local coordinates on $V(M^n)$.

Let $U \subset M^{n+p}$ be an open set and $F \in C^\infty(U)$. We call F *strictly h-convex* if $F_{ij}(u) > 0$, for all $u \in \pi^{-1}(U)$. Here :

$$F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j} - F^k_{ij} \frac{\partial F}{\partial x^k} . \tag{5.2}$$

A subset A in M^{n+p} is said to be *h-convex supporting* (in analogy with [17]) if there is an open set U in M^{n+p}, U containing A, and a strictly h-convex function $F \in C^\infty(U)$ such that its h-gradient $X^i = g^{ij} \frac{\partial F}{\partial x^j}$ has compact support in $\pi^{-1}(U)$.

The submanifold M^n is said to be *totally-geodesic* in M^{n+p} if any geodesic of the induced connection is also a geodesic of the Cartan connection of the ambient space. Cf. th. 6.2. in [7, p.1035], M^n is totally-geodesic in M^{n+p} iff $H^i_{00} = 0$, where $H^i_{00} = H^i_{ab} v^a v^b$.

Remark. Actually, th. 6.2. of [7] is formulated for the codimension one case, i.e. when M^n is a Finslerian hypersurface. It is a simple matter to refine this result in arbitrary codimension. Therefore, unlike the Riemannian case (e.g. [9]) totally-geodesic submanifolds are not characterized by the vanishing of the entire H^i_{ab}. Yet $H^i_{00} = 0$ yields $H^i_{a0} = 0$, cf. a result in [22].

Theorem C. *Let $f: M^n \longrightarrow M^{n+p}$ be a totally-geodesic immersion of the manifold M^n in the Landsberg space M^{n+p}. Then $f(M^n)$ is not contained in any of the h-convex supporting subsets of M^{n+p}.* We need to recall, cf. [22], the following :

Lemma. *If M^n is a submanifold of the Landsberg space M^{n+p} then the following formulae hold* :

$$P^1(\bar{X}, \bar{Y}) = W_{N(\bar{Y})} \bar{X}$$
$$H(\bar{X}, \bar{Y}) = (D_{\bar{Y}} N) \bar{X} + N(C(\bar{Y}, \bar{X}))$$

for any Finsler vector fields $\bar{X}$, $\bar{Y}$ on M^n. In particular, if M^n is totally-geodesic in M^{n+p} then M^n is also a Landsberg space (with the induced Finsler structure) and

$H = 0$.

The notations used to state the above lemma are those in ref. [16]. If $B \longrightarrow V(M^n)$ is the normal bundle of $f\colon M^n \longrightarrow M^{n+p}$, and $\tilde{A}_U$ is the *Weingarten operator* (corresponding to the cross-section U in B) then $W_U \bar{X} = \tilde{A}_U \gamma \bar{X}$. If H is the *second fundamental form*, then $H(\bar{X}, \bar{Y}) = \tilde{H}(\beta \bar{X}, \bar{Y})$. The *normal curvature vector* N is defined by $N(\bar{X}) = H(\bar{X}, \bar{v})$, where $\bar{v}$ is the *Liouville vector*, i.e. the cross-section in the pull-back bundle $\pi^{-1}T(M^n)$ defined by $\bar{v}(u) = (u, u)$, $u \in V(M^n)$.

The proof of our theorem C is by contradiction. Let $f(M^n)$ be contained in a h-convex supporting set and let F be a strictly h-convex function defined on some open set containing $f(M^n)$. We set $G = F \circ f$, $G_{ab} = \dfrac{\partial^2 G}{\partial u^a \partial u^b} - F^c_{ab} \dfrac{\partial G}{\partial u^c}$ and obtain :

$$G_{ab} = F_{ij} B^{ij}_{ab} + \frac{\partial F}{\partial x^i} (H^i_{ab} - H^j_{a0} B^j_b C^i_{jk}). \tag{5.3}$$

Let us contract with g^{ab} in (5.3). We obtain :

$$\Delta^h G^v = g^{ab} F_{ij} B^{ij}_{ab} \tag{5.4}$$

where $G^v = G \circ \pi$; integrating (5.4) over $V(M^n)$, with respect to the canonical Riemannian measure of the Sasaki metric (associated with the induced Finsler structure on M^n) we obtain (since supp (grad^h F^v) compact, $F^v = F \circ \pi$, yields supp (grad^h G^v) is compact, too) :

$$\int_{V(M^n)} F_{ij} g^{ab} B^i_a B^j_b * 1 = 0$$

and thus $B^i_a = 0$, a contradiction. Our theorem C is thereby completely proved.

REFERENCES

[1] AKBAR-ZADEH, H. : *Les éspaces de Finsler et certaines de leurs généralisations,* Ann. Sci. École Normale Sup., (3) **80** (1963), 1-79.

[2] AKBAR-ZADEH, H. : *Sur les sous-variétés des variétés finslériennes,* C.R. Acad. Sci. Paris, 266 (1968), 146-148.

[3] AKBAR-ZADEH, H. and BONAN, E. : *Structure presque Kaehlerienne naturelle sur le fibre tangent à une variété finslérienne,* C.R. Acad. Sci. Paris, **258** (1964), 5581-5582.

[4] AKBAR-ZADEH, H. and WEGRZYNOWSKA, A. : *Structure presque complexe associée à une connexion regulière de vecteurs,* C.R. Acad. Sci. Paris, **282** (1976), 375-377.

[5] AMICI, O., CASCIARO, B. and DRAGOMIR, S. : *On the cohomology of Finsler spaces,* Colloq. Math. Soc. "J.Bolyai", **46** (1984), 57-82.

[6] BERWALD, L. : *Ueber Finslersche und Cartansche Geometrie I. Geometrische Erklärungen der Krümmung und des Hauptskalars eines Zweidimensionalen Finslerschen Raumes,* Math. Timişoara, 17 (1941), 34-58.

[7] BROWN, M.G. : *A study of tensors which characterize a hypersurface of a Finsler space,* Canad. J. Math., 20 (1968), 1025-1036.

[8] CARTAN, E. : Les éspaces de Finsler, Act. Sei. 79, Paris, 1934, 41 p.

[9] CHEN, B.Y. : Geometry of Submanifolds, M. Dekker, Inc., New York, 1973, 298 p.

[10] CLARK, R.S. : *Evan Tom Davies,* Bull. London Math. Soc., 6 (1974), 370-376.

[11] DAVIES, E.T. : *Sulla struttura quasi complessa associata ad uno spazio di Finsler,* Atti VII Congr. U.M.I. (1963), 367.

[12] DAVIES, E.T. : *On the curvature of the tangent bundle,* Ann. Math., (4) 81 (1969), 193-204.

[13] DAVIES, E.T. and YANO, K. : *Metrics and connections in the tangent bundle,* Kodai Math. Sem. Reports, 23 (1971), 493-504.

[14] DAVIES, E.T. and YANO, K. : *On the tangent bundle of Finsler and Riemannian manifolds,* Rend. Circ. Mat. Palermo, (2) 12 (1963), 211-228.

[15] DRAGOMIR, S. : *The theorem of K. Nomizu on Finsler manifolds,* An. Univ. Timişoara, Ser. Şt. Mat., 19 (1981), 117-127.

[16] DRAGOMIR, S. : *Submanifolds of Finsler spaces,* Conf. del Sem. di Matem. dell'Univ. di Bari, 217 (1986), 1-15.

[17] GORDON, W.B. : *Convex functions and harmonic maps,* Proc. A.M.S., (2) 33 (1972), 433-437.

[18] HASSAN, B.T.M. : The theory of geodesies in Finsler spaces, Ph. D. thesis, Southampton, 1967, 108 p.

[19] HELGASON, S. : Differential Geometry and Symmetric Spaces, Acad. Press, New York, 1962, 486 p.

[20] KOBAYASHI, S. and NOMIZU, K. : Foundations of Differential Geometry, Vol. I-II, Interscience Publishers, New York, 1963, 1969.

[21] MATSUMOTO, M. : Foundations of Finsler Geometry and Special Finsler Spaces, Kasheisha Press, Kyoto, 1986.

[22] MATSUMOTO, M. : *The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry,* J. Math. Kyoto Univ., (1) 25 (1985), 107-144.

[23] MATSUMOTO, M. and TAMASSY, L. : *Scalar and gradient vector fields of Finsler spaces and holonomy groups of non-linear connections,* Demonstratio Mathematica, 13 (1980), 551-564.

[24] NUMATA, S. : *On Landsberg spaces of scalar curvature,* J. Korean Math. Soc., 12 (1975), 97-100.

[25] RUND, H. : *A divergence theorem for Finsler metrics,* Monatsh. Math., 79 (1975), 233-252.

[26] RUND, H. : *The theory of subspaces of a Finsler space, I-II.* Math. Z., 56 (1952), 363-375, 57 (1953), 193-210.

[27] RUND, H. : *Hypersurfaces of a Finsler space,* Canad. J. Math., 8 (1956), 487-503.

[28] SANINI, A. : *Applicazioni armoniche tra i fibrati tangenti di varietà Riemanniane,* Boll. U.M.I., (6) 2-A (1983), 55-63.

[29] SHIMADA, H. : *On the Ricci tensors of particular Finsler spaces,* J. Korean Math. Soc., 14 (1977), 41-63.

[30] TARINA, M. and ENGHIS, E. : *Formalisme extérieur dans la géométrie de Finsler d'un fibré vectoriel,* Proc. Nat. Sem. on Finsler Spaces, Brasov, 3 (1984), 89-97.

[31] YANO, K. and ISHIHARA, S. : Tangent and cotangent bundles, M. Dekker, Inc., New York, 1973.

[32] YANO, K. and OKUBO, T. : *On tangent bundles with Sasakian metrics of Finslerian and Riemannian manifolds,* Ann. Mat. Pura Appl., (4) 87 (1970), 137-162.

[33] VRANCEANU, G. : Lectii de Geometrie Differentialà, Vol. I-IV, Ed. Acad. R.S.R., Bucuresti, 1979.

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ÖZET

$(M^n(c), E)$, skaler eğriliği $c \neq 0$ ve karma burulma vektörü $P_j = \dot{\partial}_i N^i_j - F^i_{ji}$ sıfır olan bir Finsler uzayı olsun. $T(M^n(c)) - \{0\}$ üzerinde y^i lere göre r. dereceden pozitif homogen ve h-gradiyenti kompakt taşıyıcıya sahip olan bütün $f(x, y)$ h-harmonik fonksiyonları, $a \in IR$ olmak üzere, $f = a E^{r/2}$ ile verilebilir. Bir Finsler uzayının bir M^{n+p} Landsberg uzayı içine total-jeodezik yatırılışının resmi, M^{n+p} nin hiçbir h-konveks taşıyıcı cümlesinin içinde bulunamaz.