INDUCED RIEMANNIAN METRICS ASSOCIATED WITH THE WRONA METRIC

S. DRAGOMIR - A. FARINOLA

The metric E induced on the unit circle S^1 by the Wrona metric of IR^2 is shown to be Riemannian. The real projective line gets an induced Riemannian metric E_0 via the fibration $S^1 \to IR P^1$; we determine the geodesies of $(IR P^1, E_0)$, derive the Laplace-Beltrami operator (on functions) of E_0 and compute its positive eigen-values.

1. INTRODUCTION

Let M be an n-dimensional C^∞-differentiable connected manifold. Denote by $T(M) \to M$ the tangent bundle over M; let also $j : M \to T(M)$, $j(x) = 0_x \in T_x(M)$, $x \in M$, be the natural imbedding of M in $T(M)$ as the zero-section. We put $V(M) = T(M) - j(M)$, and denote by $\pi : V(M) \to M$ the natural projection. Then $V(M)$ is open in $T(M)$, and consequently is a $2n$-dimensional C^∞-differentiable manifold in a natural way. Let D be a subset of $T(M)$ such that $k > 0$, $u \in D$ yields $k u \in D$. Then a *Finslerian energy* on M is a map $E : D \to [0, +\infty)$ obeying the following axioms :

i) $E(u) = 0$ iff $u \in j(M)$,

ii) $E \in C^\infty(D - j(M))$, $E \in C^1(D)$ (it is assumed that $j(M) \subset D$),

iii) $E(k u) = k^2 E(u)$, for any $k > 0$, $u \in D$ (i.e. E is positively homogeneous of degree 2) ; finally, iv) if (U, x^i) are local coordinates on M and $(\pi^{-1}(U), x^i, y^i)$ are induced local coordinates on $V(M)$ then, for any $u \in D \cap \pi^{-1}(U)$ one requests $g_{ij}(u) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j E$ to be a positive definite quadratic form. Here $\dot{\partial}_i = \frac{\partial}{\partial y^i}$. A copy (M, E) is refered to as a *Finsler space*.

Eversince the basis of Finsler geometry have been settled, cf. e.g. H.RUND, [13], there have been recognized various classes of Finsler spaces, such as locally Minkowski spaces, [7, p.153], Berwald and Landsberg spaces, [7, p.160], etc. These spaces are distinguished in that they are to satisfy certain tensor identities in terms of their Cartan or Berwald connections, see [7, p.108-115]. Yet few significant explicit examples of Finsler (non-Riemannian) metrics are available in the existing literature. Among these we may mention *Randers' metric*, cf. [12], [10], *Berwald-Moor's metric*, cf. [8], etc.; both are only semi-definite and lead to applications in general relativity. Also we should cite K.Okubo's method, cf. [7, p.108], to produce Finsler metrics with a prescribed indicatrix. About fifty years ago, W.WRONA, cf. [16, p.281], has proposed the following Finsler metric in the real plane IR^2 : The distance between two points P, Q of IR^2 is defined to be the number $\frac{PQ}{OS}$ where PQ, OS are Euclidean measures of length and OS is the perpendicular from a fixed origin O in IR^2 to the segment PQ. The quantity $\frac{PQ}{OS}$ is to measure the "length" of the segment PQ as the direction of PQ varies, i.e. in a "Finslerian" way; clearly, any line passing through O has no length measure defined on it. The example of W.Wrona has been further generalized by S.HOJO, [6]. In the present paper, if L is the Wrona metric of IR^2, we consider the metric induced by L on the standard unit circle S^1; this appears to be Riemannian. Consequently the real projective line $IR\,P^1$ gets a naturally induced (via the fibration $S \to IR\,P^1$) Riemannian structure E_0. An explicit determination of the geodesies of $(IR\,P^1, E_0)$ is performed. As an application of this we derive the Laplace-Beltrami operator associated with E_0, i.e.

$$\Delta f = -(1+\eta^2)^2 \left[\frac{d^2 f}{d\eta^2} + 2 \frac{\eta}{1+\eta^2} \frac{df}{d\eta} \right] \tag{1.1}$$

where η stands for the local coordinate on $IR\,P^1$. As an attempt to compute Spec $(IR\,P^1, E_0)$ we find:

$$\lambda_n = \left[\frac{2n\pi \pm \alpha}{\frac{\pi}{2} - \varepsilon} \right]^2, \quad n \in \mathbf{Z} \tag{1.2}$$

whenever $\lambda > 0$, for some $\alpha > 0$, $\varepsilon > 0$, i.e. one determines the positive eigen-values of (1.1).

2. THE INDUCED METRIC ON S^1

Let S^1 be the unit circle in IR^2; let (x, y) be cartesian coordinates on IR^2 and $(x, y, \dot{x}, \dot{y})$ the induced coordinates on $T(IR^2) = IR^4$. We set $D = \{(x, y, \dot{x}, \dot{y}) \mid x\dot{y} - y\dot{x} \neq 0\}$; then IR^2 carries the Finsler metric $L : D \to [0, +\infty)$ given by:

$$L(x, y, \dot{x}, \dot{y}) = \frac{\dot{x}^2 + \dot{y}^2}{|x\dot{y} - y\dot{x}|}, \tag{2.1}$$

i.e. in the terminology of our § 1, $E = L^2$ is a Finsler energy on IR^2. Then (2.1) is refered to as the *Wrona metric* of the plane. See [16] and [7, p.107].

Generally, if $(M, \bar{E})$ is a Finsler space and $\psi : N \to M$ an immersion of a given differentiable manifold N in M, then N carries an *induced Finsler energy* $E = \bar{E} \circ \psi_*$, where ψ_* denotes the differential of ψ. Thus N turns into a Finsler space itself, and it is an open problem to classify the immersions ψ for which E is Riemannian. In order to compute $L \circ i_*$, where $i : S^1 \to IR^2$ denotes the canonical inclusion, we cover S^1 with the atlas $\{(U^+,h^+),(U^-,h^-),(L_+,h_+),(U_-,h_-)\}$, where $U^+ = \{(x, \sqrt{1-x^2}) \mid -1 < x < 1\}$, $U_- = \{(x, \sqrt{1-x^2}) \mid -1 < x < 1\}$, $U_+ = \{(\sqrt{1-y^2}, y) \mid -1 < y < 1\}$, $U_- = \{(-\sqrt{1-y^2}, y) \mid -1 < y < 1\}$, while $h_+ : U^+ \to IR, h^- : U^- \to IR$, $h_+ : U_+ \to IR$ and $h_- : U_- \to IR$ are given by $h^+ = p_1|_{U^+}$, $h^- = p_1|_{U^-}$, $h_+ = p_2|_{U_+}$, $h_- = p_2|_{U_-}$ where $p_i : IR^2 \to IR$ stand for the natural projections. Let $\pi_1 : T(IR^2) \to IR^2$ be the natural projection. Then $T^+(S^1) = \pi_1^{-1}(U^+)$, $T^-(S^1) = \pi_1^{-1}(U^-)$, $T_+(S^1) = \pi_1^{-1}(U_+)$, $T_-(S^1) = \pi_1^{-1}(U_-)$ is an open cover of $T(S^1)$. Tangent vectors X on S^1 at $(x, y) \in S^1$ are precisely those $X \in T_{(x,y)}(IR^2)$ with $X(F) = 0$, where $F(x, y) = x^2 + y^2 - 1$. Suppose $(x, y) \in U^+$; then $T_{(x,y)}(S^1)$ is spanned by $\left.\frac{\partial}{\partial x}\right|_{(x,y)} - \frac{x}{y}\left.\frac{\partial}{\partial y}\right|_{(x,y)}$, $y = \sqrt{1-x^2}$, $-1<x<1$. Let $\{e_A\}_{1 \leq A \leq 4}$ be the canonical linear basis of IR^4. Let us identify $\left.\frac{\partial}{\partial x}\right|_{(x,y)}$, $\left.\frac{\partial}{\partial y}\right|_{(x,y)}$ with e_3, e_4 respectively. Therefore $T^+(S^1) = \left\{\left(x, \sqrt{1-x^2}, \lambda, -\frac{x}{\sqrt{1-x^2}}\lambda\right) \mid -1 < x < 1, \lambda \in IR\right\}$. Set $IR^* = IR - \{0\}$. We set $V(S^1) = T(S^1) - 0$. Let then π be the restriction of π_1 to $V(S^1)$. Therefore $V^+(S^1) = \pi^{-1}(U^+)$, $V^-(S^1) = \pi^{-1}(U^-)$, $V_+(S^1) = \pi^{-1}(U_+)$, $V_-(S^1) = \pi^{-1}(U_-)$ are chart domains of an atlas of $V(S^1)$ as follows: We put $H^+ : V^+(S^1) \to IR^2$, $H^+\left(x, \sqrt{1-x^2}, \lambda, -\frac{x}{\sqrt{1-x^2}}\lambda\right) = (x, \lambda)$, $-1 < x < 1$, $\lambda \in IR^*$. Clearly $V^+(S^1) = T^+(S^1) \cap D$. Let $E = L^2$ be the energy function associated with the Lagrangian (2.1). Let E^+ be the local expression of $E|_{V(S^1)}$ with respect to the local chart $(V^+(S^1), H^+)$. Then $E^+(x, \lambda) = \frac{\lambda^2}{1-x^2}$. The reader might establish, as an exercise, the similar expressions of E^-, E_+, E_- (definitions are obvious). Note that $\frac{d^2 E^+}{d\lambda^2}$ is a function of positional arguments only, i.e. *the metric induced by* (2.1) *on* S_1 *is Riemannian.*

Let $IR\,P^1$ be the real projective line. We cover $IR\,P^1$ with the atlas $\{(U_j, \psi_j)\}_{j=1,2}$. Here U_j consists of all lines $L_\xi = \{t\xi \mid t \in IR^*\}$, $\xi \in S^1$, $\xi = (\xi^1, \xi^2)$, $\xi^j \neq 0$. Consider for instance $\psi_2 : U_2 \to IR$, $\psi_2(L_\xi) = \frac{\xi^1}{\xi^2} = \eta$. Next we consider the fibration $p : S^1 \to IR\,P^1$ and denote by $\underline{p}$ the local expression of p, i.e. $\underline{p}:(-1\,,1)\to IR$, $\underline{p}=\psi_2\circ p\circ(h^+)^{-1}$. Note that $\underline{p}(x)=x(1-x^2)^{-1/2}$ and its Jacobian is given by $d\underline{p}/dx=(1-x^2)^{-3/2}$.

3. THE INDUCED METRIC ON IR P^1

Let $\xi_0\in S^1$, $P_0=L_{\xi_0}\in IR\,P^1$. If $\xi_0\in U^+$, $\xi_0=\left(x_0,\sqrt{1-x_0^2}\right)$, let $x:(-\varepsilon,\varepsilon)\to(-1,1)$ be a differentiable function, such that $x(0)=x_0$. Then $C:(-\varepsilon,\varepsilon)\to U^+$, $C(t)=(x(t),\sqrt{1-x(t)^2})$, $|t|<\varepsilon$, is a curve in S^1 with $C(0)=\xi_0$. Therefore $a:(-\varepsilon,\varepsilon)\to U_2$, $a(t)=L_{C(t)}$, is a curve in $IR\,P^1$ with $a(0)=P_0$. Next, we need to determine the tangent space $T_{P_0}(IR\,P^1)$. Let $a(t)\in U_2$, $|t|<\varepsilon,\varepsilon>0$, be a curve; let $\underline{a}=\psi_2\circ a$ be its local expression. Note that $\underline{a}(t)=x(t)(1-x(t)^2)^{-1/2}$, $|t|<\varepsilon$. Let $C(x_0)$ be the space of all smooth functions $x:(-\varepsilon,\varepsilon)\to(-1,1)$ with $x(0)=x_0$. Then we obtain $T_{P_0}(IR\,P^1)=\left\{\frac{dx}{dt}(0)(1-x_0^2)^{-3/2}\frac{\partial}{\partial\eta}\Big|_{P_0}\ ;\ x\in C(x_0)\right\}$. We set $V(IR\,P^1)=T(IR\,P^1)-\{0\}$ and denote by $\mathrm{p}:V(IR\,P^1)\to IR\,P^1$ the natural projection. Let $(\mathrm{p}^{-1}(U_2),\phi_2)$ be the local chart induced on $V(IR\,P^1)$ by (U_2,ψ_2), i.e. $\phi_2:\mathrm{p}^{-1}(U_2)\to IR^2$ is given by

$$\phi_2\left(\frac{dx}{dt}(0)(1-x_0^2)^{-3/2}\frac{\partial}{\partial\eta}\Big|_{P_0}\right)=\left(x_0(1-x_0^2)^{-1/2},\frac{dx}{dt}(0)(1-x_0^2)^{-3/2}\right)=(\eta\,,\dot\eta).$$

Let $\underline{dp}$ be the local expression of the differential dp, i.e. the following diagram is commutative:

$$\begin{array}{ccc} V^+(S^1) & \xrightarrow{\ dp\ } & \mathrm{p}^{-1}(U_2) \\ \Big\downarrow H^+ & & \Big\downarrow \phi_2 \\ (x\,,\lambda)\in IR^2 & \xrightarrow[\ \underline{dp}\ ]{} & IR^2\ni(\eta\,,\dot\eta) \end{array}$$

Note that $\underline{dp}(x,\lambda)=(x[1-x^2]^{-1/2},\lambda[1-x^2]^{-3/2})$, $|x|<1$, $\lambda\in IR^*$. Clearly $\underline{dp}$ is an isomorphism on the fibres; its inverse might be locally written:

$$x=\eta(1+\eta^2)^{-1/2},\quad \lambda=\dot\eta(1+\eta^2)^{-3/2}. \tag{3.1}$$

We may consider the energy function:

$$E_0^+(\eta\,,\dot\eta)=E^+(x\,,\lambda) \tag{3.2}$$

where $x\,,\lambda$ are given by (3.1). Therefore $IR\,P^1$ carries the Riemannian metric locally expressed by $E_0^+(\eta,\dot\eta)=\dot\eta^2(1+\eta^2)^{-2}$. The associated metric tensor has the (local) component $g^+(\eta)=(1+\eta^2)^{-2}$. Thus the (local) component of the

Levi-Civita connection of $(IR\,P^1, E_0)$ might be computed from $\Gamma^+(\eta) = \frac{1}{g^+}\gamma^+(\eta)$, where $\gamma^+(\eta) = \frac{1}{2}\frac{\partial g^+}{\partial \eta}$ is the Christoffel symbol of the first kind; one obtains $\Gamma^+(\eta) = -2\eta(1+\eta^2)^{-1}$.

At this point we may find all the geodesics of $(IR\,P^1, E_0)$; the equation of the geodesics reads $\frac{d^2 \underline{a}}{dt^2} + \Gamma^+[\underline{a}(t)]\left(\frac{d\underline{a}}{dt}\right)^2 = 0$ and this is equivalent to :

$$\frac{d^2 x}{dt^2} + \frac{x}{1-x^2}\left(\frac{dx}{dt}\right)^2 = 0. \tag{3.3}$$

The initial conditions $a(0) = P_0$, $\frac{da}{dt}(0) = X_0$, $X_0 \in p^{-1}(L_{\xi_0})$, $\xi_0 \in U^+$, or $\underline{a}(0) = x_0(1-x_0^2)^{-1/2}$, $\frac{d\underline{a}}{dt}(0) = X_0$, where $X_0 = \underline{X}_0 \left.\frac{\partial}{\partial\eta}\right|_{P_0}$, furnish :

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = \underline{X}_0(1-x_0^2)^{3/2}. \tag{3.4}$$

Integration of (3.3) with the initial data (3.4) leads to :

$$x(t) = \sin[\underline{X}_0(1-x_0^2)\,t + \arcsin x_0], \quad |t| < \varepsilon \tag{3.5}$$

for some $\varepsilon > 0$.

4. COMPUTING Spec $(IR\,P^1, E_0)$

Let M be a Riemannian manifold and $\Delta : C^\infty(M) \to C^\infty(M)$ its Laplace-Beltrami operator (on functions). Let $x \in M$ and $f \in C^\infty(M)$; let also $\{X_i\}_{1 \leqq i \leqq n}$ be an orthonormal basis in $T_x(M)$, where $n = \dim(M)$. Next we consider the geodesies a_i, $1 \leqq i \leqq n$, of M determined by the initial data (x, X_i), respectively. We shall use the well known formula :

$$(\Delta f)(x) = -\sum_{i=1}^{n} \frac{d^2}{dt^2}[f \circ a_i]\,|_{t=0}. \tag{4.1}$$

As a consequence of (4.1), the Laplace operator $\Delta : C^\infty(IR\,P^1) \to C^\infty(IR\,P^1)$ associated with E_0 is expressed by :

$$(\Delta f)(P_0) = -\left.\frac{d^2}{dt^2}[f \circ a]\right|_{t=0} \tag{4.2}$$

where a is the geodesic of $(IR\,P^1, E_0)$ with the initial data (P_0, X_0). Since X_0 must be a unit tangent vector (with respect to $g^+(P_0)$) its component $\underline{X}_0$ is expressed by $\underline{X}_0 = (1-x_0^2)^{-1}$. Therefore, by (3.5), the geodesic a might be written :

$$\underline{a}(t) = \tan(t + t_0), \quad |t| < \varepsilon \tag{4.3}$$

where $t_0 = \arcsin x_0$. Then (4.2) transforms into

$$\Delta f = -[\cos t_0]^{-4}\left[\frac{d^2 f}{d\eta^2} + \sin(2t_0)\frac{df}{d\eta}\right],$$

or, after some computation, one obtains the formula (1.1), for any $f \in C^\infty(IR P^1)$. Now the problem $\Delta f = \lambda f$ is equivalent to:

$$f''(x) + \frac{2x}{1+x^2} f'(x) + \frac{\lambda}{(1+x^2)^2} f(x) = 0. \tag{4.4}$$

Towards the self-adjoint form of (4.4) one substitutes $t = \arctan x$, $y(t) = f(x)$; this procedure yields:

$$\frac{d^2 y}{dt^2} + \lambda y = 0, \quad |t| < \frac{\pi}{2} - \varepsilon, \quad \varepsilon > 0. \tag{4.5}$$

Finally, cf. e.g. [11], the general solution of (4.5) is $y(t) = c_1 t + c_2$ if $\lambda = 0$, $y(t) = c_1 \cos(\sqrt{\lambda}\, t) + c_2 \sin(\sqrt{\lambda}\, t)$ if $\lambda > 0$, and $y(t) = c_1 \operatorname{ch}(\sqrt{-\lambda}\, t) + c_2 \operatorname{sh}(\sqrt{-\lambda}\, t)$ if $\lambda < 0$, where $c_1, c_2 \in IR$. If one assigns the boundary conditions $y\left(-\frac{\pi}{2} + \varepsilon\right) = A$, $y\left(\frac{\pi}{2} - \varepsilon\right) = B$, then, provided for instance that $\lambda > 0$, one obtains $\lambda = \lambda_n$, $n \in \mathbf{Z}$, where λ_n is given by our (1.2) while $\alpha = \arccos k$, $k = \frac{A+B}{2c_1}$, $c_1 \neq 0$, $|k| < 1$.

BIBLIOGRAPHY

[1] AKBAR-ZADEH, H. : *Les espaces de Finsler et certaines de leurs généralisations*, Ann. Sci. Ecole Norm. Sup. (3) **80** (1963), 1-79.

[2] DRAGOMIR, S. : *On Finsler manifolds with complete horizontal leaves*, Seminarberichte, Fachbereich Mathematik und Informatik, **25** (1986), 39-55.

[3] DRAGOMIR, S. : *Submanifolds of Finsler spaces*, Conferenze del Seminario di Matem. dell'Univ. di Bari 217 (1986), 1-15.

[4] DRAGOMIR, S. and IANUS, S. : *On the holomorphic sectional curvature of Kaehlerian Finsler spaces, I-II*, Tensor, N.S. 39 (1982), 95-98, Rendiconti di Matem. (4) 3 (1983), 757-763.

[5] GRIFONE, J. : *Structure presque tangente et connexions, I-II*, Ann. Inst. Fourier (1) **22** (1972), 287-334.

[6] HOJO, S. : *On geodesics of certain Finsler metrics*, Tensor, N.S. 34 (1980), 211-217.

[7] MATSUMOTO, M. : **Foundations of Finster geometry and special Finsler spaces**, Kasheisha Press, Kyoto, 1986.

[8] MATSUMOTO, M. and SHIMADA, H. : *On Finsler spaces with 1-form metric, II. Berwald-Moor's metric* $L = (y^1 y^2 \dots y^n)^{1/n}$, Tensor, N.S. 32 (1978), 275-278.

[9] MISHRA, R.S. and PANDE, H.D. : *Conformai identities*, İst. Üniv. Fen Fak. Mec. Seri A, 31 (1966), 39-48.

[10] NUMATA, S. : *On the torsion tensors* R_{hjk} *and* P_{hjk} *of Finsler spaces with a metric* $ds = (g_{ij}(dx)\, dx^i dx^j)^{1/2} + b_i(x)\, dx^i$, Tensor, N.S. 32 (1978), 27-31.

[11] RABENSTEIN, A.L. : **Introduction to ordinary differential equations**, Acad. Press, New York, 1972.

[12] RANDERS, G. : *On an asymmetric metric in the four-space of general relativity*, Phys. Rev., (2) 59 (1941), 195-199.

[13] RUND, H. : **The differential geometry of Finsler spaces**, Springer Verlag, Berlin, 1959.

[14] SHIBATA, C. : *On Finsler spaces with Kropina metric*, Rep. Math. Phys., 13 (1978), 117-128.

[15] SHIMADA, H. : *On Finsler spaces with the metric*
$$L = \sqrt[m]{a_{i_1 i_2 \cdots i_m}(x)\, y^{i_1} y^{i_2} \dots y^{i_m}},$$
Tensor, N.S. 33 (1979), 365-372.

[16] WRONA, W. : *Neues Beispiel einer Finslerschen Geometrie*, Prace Mat. Fiz., 46 (1938), 281-290.

UNIVERSITA DEGLI STUDI DI BARI
DIPARTIMENTO DI MATEMATICA
VIA G.FORTUNATO, CAMPUS UNIVERSITARIO
70125 BARI, ITALY

ÖZET

Bu çalışmada, S^1 birim dairesi üzerinde IR^2 nin Wrona metriği yardımıyla oluşturulan E metriğinin, Riemann metriği olduğu ispat edilmektedir.