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THE GEOMETRIC INTERPRETATION OF THE SECTIONAL CURVATURE OF A FINSLER SPACE

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Given a generalized Finsler space M the manifold $V(M) = T(M) - 0$ of all tangent directions on M admits a naturally induced pseudo-Riemannian structure. Also, there is a linear connection on $V(M)$ corresponding to the Miron connection [9] of M; in terms of the associated exponential formalism on $V(M)$ the following geometric interpretation of the vertical sectional curvature s occurs: if p is a Finslerian 2-plane on M then $s(p)$ approximates the difference between the length of a circumference centred at the origin in p and the length of its exponential projection on $V(M)$.

1. NOTATIONS, CONVENTIONS AND BASIC FORMULAE

Let M be an n-dimensional C^∞-differentiable manifold and $\pi : V(M) \to M$ the natural projection, where $V(M) = T(M) - 0$, while $T(M) \to M$ stands for the tangent bundle over M. Let $\pi^{-1} T(M) \to V(M)$ be the pullback bundle of $T(M)$ by π. This is a real differentiable vector bundle of rank n.

A *generalized metrical Finsler structure* on M is a non-degenerated symmetric Finsler (0,2)-tensor field g, $g \in \Gamma(V(M), \pi^{-1} T^*(M) \otimes \pi^{-1} T^*(M))$. Throughout, if $E \to N$ is a given vector bundle over the manifold N, then $\Gamma(N, E)$ denotes the module (over the ring $C^\infty(N)$ of all real valued smooth functions on N) of all smooth cross-sections in E. A pair (M, g) is a *generalized Finsler space,* cf. R.MIRON, [9]. A *non-linear connection* on $V(M)$ is a differential system $N : u \to N_u \subset T_u(V(M))$ on $V(M)$ such that:

$$T_u(V(M)) = N_u \oplus \operatorname{Ker}(d_u \pi) \tag{1.1}$$

for each *tangent direction* $u \in V(M)$ on M. See W.BARTHEL, [1]. Consequently $(V(M), N)$ turns to be a non-holonomic space, in the sense of G.VRANCEANU, [10].

Next we consider the bundle epimorphism L given by $L: T(V(M)) \to \pi^{-1} T(M)$, $L_u \tilde{X} = (u, (d_u \pi) \tilde{X})$, for any $u \in V(M), \tilde{X} \in T_u(V(M))$. Note that $\operatorname{Ker}(L) = \operatorname{Ker}(d\pi)$; thus, if some nou-linear connection N on $V(M)$ is fixed, each $L_u : N_u \to \pi_u^{-1} T(M)$ is a IR-linear isomorphism, where $\pi_u^{-1} T(M) = \{u\} \times T_{\pi(u)}(M)$ denotes the fibre over u in $\pi^{-1} T(M)$. We set $\beta_u \equiv (L|_{N_u})^{-1}$, $u \in V(M)$. The resulting bundle isomorphism $\beta : \pi^{-1} T(M) \to N$ is refered to as the *horizontal lift* associated with N.

Let (U, x^i) be a local coordinate system on M and let $(\pi^{-1}(U), x^i, y^i)$ be the induced local coordinates on $V(M)$. Locally, cf. [1], a non-linear connection N on $V(M)$ is given by a Pfaffian system:

$$\delta y^i \equiv d y^i + N^i_j (x, y) \, d x^j = 0 . \tag{1.2}$$

To state this in modern language, let $X_i : \pi^{-1}(U) \to \pi^{-1} T(M)$, $X_i(u) = \left(u, \frac{\partial}{\partial x^i}\Big|_{\pi(u)} \right)$, for any $u \in \pi^{-1}(U)$. Next, let us set $\delta_i = \beta X_i, 1 \leqq i \leqq n$. Let us put $\partial_i = \frac{\partial}{\partial x^i}$, $\dot{\partial}_i = \frac{\partial}{\partial y^i}$ for simphcity. Then there exists a uniquely determined system of n^2 smooth functions $N^i_j \in C^\infty(\pi^{-1}(U))$ such that $\delta_i = \partial_i - N^j_i \dot{\partial}_j$ and N^i_j are usually termed the *coefficients* of the non-linear connection N with respect to (U, x^i). Now (1.2) means that, for any $u \in \pi^{-1}(U)$, N_u is spanned by $\{\delta_i|_u\}_{1 \leqq i \leqq n}$ over the reals.

The *vertical lift* is the bundle isomorphism γ defined by $\gamma : \pi^{-1} T(M) \to \mathrm{Ker}(d\pi)$, $\gamma(X_i) = \dot{\partial}_i$. The definition of γ does not depend upon the choice of local coordinates.

Let $P_{1,u}, P_{2,u}$ be the natural projections associated with the direct sum decomposition (1.1). We shall need the bundle morphisms:

$$P_3 = \gamma \circ L \; , \quad P_4 = \beta \circ G \tag{1.3}$$

where $G : T(V(M)) \to \pi^{-1} T(M)$ denotes the *Dombrowski mapping*, i.e. $G_u \tilde{X} = \gamma_u^{-1} \tilde{X}_v$, where $\tilde{X}_v = P_{2,u} \tilde{X}, \tilde{X} \in T_u(V(M))$, $u \in V(M)$. Cf. P.DOMBROWSKI, [3].

Let (M, g) be a generalized Finsler space. Each fibre $\pi_u^{-1} T(M), u \in V(M)$, of the pullback bundle carries a semi-definite inner product g_u and $u \to g$ is smooth. Therefore $\pi^{-1} T(M) \to V(M)$ turns into a pseudo-Riemannian vector bundle. Moreover $V(M)$ admits the pseudo-Riemannian metric:

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(L\tilde{X}, L\tilde{Y}) + g(G\tilde{X}, G\tilde{Y}) \tag{1.4}$$

for any $\tilde{X}, \tilde{Y} \in \Gamma(V(M), T(V(M)))$ and some fixed non-linear connection N on $V(M)$ (with respect to which the Dombrowski map G is derived). If g is positive-definite then $(V(M), \tilde{g})$ turns to be a $2n$-dimensional smooth Riemannian manifold.

Let ∇ be a connection in the pullback bundle $\pi^{-1}T(M)$ of a given generalized Finsler space (M, g). In contrast with the general situation of a connection in an arbitrary vector bundle, given a non-linear connection N on $V(M)$, two concepts of torsion might be associated with ∇ :

$$\begin{aligned}\tilde{T}(\tilde{X}, \tilde{Y}) &= \nabla_{\tilde{X}} L \tilde{Y} - \nabla_{\tilde{Y}} L \tilde{X} - L[\tilde{X}, \tilde{Y}] \\ \tilde{T}_1(\tilde{X}, \tilde{Y}) &= \nabla_{\tilde{X}} G \tilde{Y} - \nabla_{\tilde{Y}} G \tilde{X} - G[\tilde{X}, \tilde{Y}]\end{aligned} \tag{1.5}$$

for any tangent vector fields $\tilde{X}, \tilde{Y}$ on $V(M)$. Nevertheless, note that only the definition of $\tilde{T}_1$ depends on the choice of N. Next we consider :

$$T(X, Y) = \tilde{T}(\beta X, \beta Y), \quad S^1(X, Y) = \tilde{T}_1(\gamma X, \gamma Y) \tag{1.6}$$

for any $X, Y \in \Gamma(V(M), \pi^{-1} T(M))$. We shall need the following result, cf. [9]:

Theorem 1.1. *There exists a unique connection ∇ in the pullback bundle $\pi^{-1} T(M)$ of the generalized Finsler space (M, g, N) such that the following axioms are satisfied:*

$$\nabla g = 0 \tag{1.7}$$

$$T = 0, \quad S^1 = 0. \tag{1.8}$$

Moreover ∇ is expressed by:

$$\begin{aligned}2g(\nabla_{\beta X} Y, Z) = {} & g(Z, L[\beta X, \beta Y]) - g(X, L[\beta Y, \beta Z]) - \\ & - g(Y, [\beta X, \beta Z]) - (\beta X)(g(Y, Z)) - \\ & - (\beta Y)(g(Z, X)) + (\beta Z)(g(X, Y))\end{aligned} \tag{1.9}$$

$$\begin{aligned}2g(\nabla_{\gamma X} Y, Z) = {} & g(Z, G[\gamma X, \gamma Y]) - g(X, G[\gamma Y, \gamma Z) - \\ & - g(Y, G[\gamma X, \gamma Z]) - (\gamma X)(g(Y, Z)) - \\ & - (\gamma Y)(g(Z, X)) + (\gamma Z)(g(X, Y))\end{aligned} \tag{1.10}$$

for any $X, Y, Z \in \Gamma(V(M), \pi^{-1} T(M))$.

Next we consider the linear connection $\tilde{\nabla}$ on $V(M)$ defined by :

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \beta \nabla_{\tilde{X}} L \tilde{Y} + \gamma \nabla_{\tilde{X}} G \tilde{Y} \tag{1.11}$$

where ∇ is the connection in $\pi^{-1} T(M)$ furnished by Theorem 1.1. The following result holds :

Theorem 1.2. *Let (M, g) be a generalized Finsler space carrying the non-linear connection N. Then the linear connection* (1.11) *is subject to :*

$$\tilde{\nabla} \tilde{g} = 0. \tag{1.12}$$

$$\tilde{\nabla} P_j = 0, \quad j \in \{1, 2, 3, 4\}. \tag{1.13}$$

If $\tilde{A}$ is the torsion 2-form of $\tilde{\nabla}$ then :

$$\widetilde{A}(\widetilde{X}, \widetilde{Y}) = \beta\, \widetilde{T}(\widetilde{X}, \widetilde{Y}) + \gamma\, \widetilde{T}_1(\widetilde{X}, \widetilde{Y}) \tag{2.14}$$

for any tangent vector fields $\widetilde{X}, \widetilde{Y}$ *on* $V(M)$.

The proof of Theorem 1.2, being straightforward, is left as an exercise to the reader.

2. EXPONENTIAL FORMALISM ON A GENERALIZED FINSLER SPACE

Let (M, g) be a generalized Finsler space carrying the non-linear connection N. Consider the linear connection (1.11) on the pseudo-Riemannian manifold $(V(M), \widetilde{g})$. Let $u_0 \in V(M)$ be a fixed tangent direction on M. Let :

$$\exp_{u_0} : W_{\widetilde{0}} \to W_{u_0} \tag{2.1}$$

be the exponential mapping associated with the linear connection (1.11), where $W_{\widetilde{0}}$ and W_{u_0} are suitable chosen open neighborhoods of the zero tangent vector $\widetilde{0}$ in $T_{u_0}(V(M))$, and of u_0 in $V(M)$, respectively. On the other hand, for any Finsler space M, there is an exponential formalism associated with the Cartan connection of M, such as developed in B.T.HASSAN, [7]. This might be related to (2.1) as follows: Let $E : T(M) \to [0, +\infty)$ be a fixed Finsler energy on M. If the generalized Finsler metric g is positive-definite and its (local) components are subject to $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j E$, then (M, g) is a *Finsler space*. Moreover suppose that N is (locally) given by :

$$N^i_j = \frac{1}{2}\, \dot{\partial}_j \gamma^i_{00} \tag{2.2}$$

where :

$$\gamma^i_{00} = \gamma^i_{jk}\, y^j y^k \quad , \quad \gamma^i_{jk} = g^{ih} |\, jk\, , h\,|$$

$$|\, jk\, , h\,| = \frac{1}{2} (\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk}) \,.$$

Then the *Miron connection* (1.9)-(1.10) coincides with the unique regular Cartan connection of (M, E), such as introduced in E.CARTAN, [2].

Let $x_0 = \pi(u_0)$, $x_0 \in M$. Put next $L(u) = E(u)^{1/2}$, for any $u \in V(M)$. We shall use the following, [7] :

Theorem 2.1. *Let* (M, E) *be a Finsler space and* ∇ *its Cartan connection. Then there exists* $\varepsilon > 0$ *such that the following second order ordinary differential system :*

$$\nabla_{\frac{d\tilde{C}}{dt}} L \frac{d\tilde{C}}{dt} = 0 \tag{2.3}$$

admits a unique solution $C = C_{X_0}$, $C_{X_0} : (-2, 2) \to M$ *satisfying the initial conditions* $C_{X_0}(0) = x_0$, *and* $\frac{dC_{X_0}}{dt}(0) = X_0$, $X_0 \in T_{x_0}(M)$, *provided that* $L(X_0) < \varepsilon$.

To make the notation in (2.3) clear, we mention that given a regular curve $C : I \to M$, for some open interval $I \subset IR$, one denotes by $\tilde{C} : I \to V(M)$ the *natural lift* of C, i.e. $\tilde{C}(t) = \frac{dC}{dt}(t)$, $t \in I$. We shall need the following :

Theorem 2.2. *The natural lift* $\tilde{C}$ *of any solution* C *of* (2.3), *i.e. of any geodesic of the Finsler space* (M, E), *is a horizontal auto-parallel curve of the linear connection* (1.11). *That is* :

$$\tilde{\nabla}_{\frac{d\tilde{C}}{dt}} \frac{d\tilde{C}}{dt} = 0 \tag{2.4}$$

$$\frac{d\tilde{C}}{dt}(t) \in N_{\tilde{C}(t)} \tag{2.5}$$

for any value of the parameter t.

See [4]. There is $\delta > 0$ such that the open set :

$$\{ \tilde{X} \in T_{u_0}(V(M)) \mid \tilde{g}_{u_0}(\tilde{X}, \tilde{X})^{1/2} < \delta \}$$

is contained in $W_{\tilde{0}}$. If $u_0 \in V(M)$ is chosen such that $L(u_0) < \varepsilon$, then according to Theorem 2.1., there is a unique solution $C_{u_0} : (-2, 2) \to M$ of (2.3) with initial data (x_0, u_0). We may put :

$$\exp_{x_0} u_0 = C_{u_0}(1) . \tag{2.6}$$

By our Theorem 2.2. the natural lift $\tilde{C}_{u_0}$ of C_{u_0} is a solution of (2.4). Note also that $\tilde{C}_{u_0}(0) = u_0$. Next we set $\tilde{X}_0 = \frac{d\tilde{C}_{u_0}}{dt}(0)$, $\tilde{X}_0 \in T_{u_0}(V(M))$. Let $\mathrm{p} = \min(\varepsilon, \delta) > 0$. We establish firstly the following :

Lemma 2.1. *If* $L(u_0) < \mathrm{p}$ *then* $\tilde{X}_0 \in W_{\tilde{0}}$.

Proof. It is enough to prove that $\tilde{g}_{u_0}(\tilde{X}_0, \tilde{X}_0)^{1/2} < \mathrm{p}$. Let ν be the Liouville vector field on M, i.e. $\nu \in \Gamma(V(M), \pi^{-1} T(M))$, $\nu(u) = (u, u)$, $u \in V(M)$. We use now the property (2.5) of $\tilde{C}_{u_0}$ and the definition (1.4). By the classical Euler theorem on positively homogeneous functions one has :

$$\tilde{g}_{u_0}(\tilde{X}_0, \tilde{X}_0) = g_{u_0}\left(L\frac{d\tilde{C}_{u_0}}{dt}(0), L\frac{d\tilde{C}_{u_0}}{dt}(0)\right) =$$
$$= g_{u_0}\left(v\left(\frac{dC_{u_0}}{dt}(0)\right), v\left(\frac{dC_{u_0}}{dt}(0)\right)\right) =$$
$$= g_{u_0}(v(u_0), v(u_0)) = E(u_0)$$

and the proof is complete.

By our Lemma 2.1., if $L(u_0) < \rho$ then :

$$\exp_{u_0}\tilde{X}_0 = \tilde{C}_{u_0}(1)\,. \tag{2.7}$$

Therefore, the link between the exponentials (2.6) - (2.7) is expressed by :

$$\pi(\exp_{u_0}\tilde{X}_0) = \exp_{x_0} u_0\,. \tag{2.8}$$

3. SECTIONAL CURVATURE OF GENERALIZED FINSLER SPACES

Let (M, g) be a generalized Finsler space. Suppose from now on that g is positive-definite. The 2-dimensional linear subspaces of the fibres of the pullback bundle $\pi^{-1}T(M)$ give rise to a bundle $GF_2(M)$ over $V(M)$, with projection $\mathrm{p} : GF_2(M) \to V(M)$ and standard fibre the Grassman manifold $G_{2,n}$ of all 2-planes in IR^n. The synthetic object $GF_2(M)$ $(V(M), \mathrm{p}, G_{2,n})$ is called the Finsler-Grassmann bundle of M. Let $u_0 \in V(M)$ be a fixed tangent direction on M and $p \in GF_2(M)$, $\mathrm{p}(p) = u_0$. Let N be a non-hnear connection on $V(M)$ and β the corresponding horizontal lift. Let $\tilde{\mathrm{p}} : G_2(V(M)) \to V(M)$ be the Grassmann bundle of all 2-planes tangent to $V(M)$. We set $\gamma(p) = \{\gamma X \mid X \in p\}$, and $\beta(p) = \{\beta X \mid X \in p\}$. Then $\gamma(p)$, $\beta(p) \in G_2(V(M))$. Moreover, if $\{X, Y\}$ is an orthonormal basis of p (with respect to g_{u_0}) then $\{\gamma X, \gamma Y\}$, $\{\beta X, \beta Y\}$ are basis in $\gamma(p)$, $\beta(p)$ respectively (orthonormal with respect to the inner product $\tilde{g}_{u_0}$). Let $\tilde{B}$ be the curvature 2-form of the linear connection (1.11). As a consequence of (1.12) the (0,4)-tensor field $\tilde{B}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = \tilde{g}(\tilde{B}(\tilde{Z}, \tilde{W})\tilde{Y}, \tilde{X})$ verifies :

$$\begin{aligned} &\tilde{B}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) + \tilde{B}(\tilde{X}, \tilde{Y}, \tilde{W}, \tilde{Z}) = 0 \\ &\tilde{B}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) + \tilde{B}(\tilde{Y}, \tilde{X}, \tilde{Z}, \tilde{W}) = 0\,. \end{aligned} \tag{3.1}$$

Since (3.1) holds, we may consider the (well-defined) map $b : G_2(V(M)) \to IR$, $b(\tilde{p}) = \tilde{B}_u(\tilde{X}, \tilde{Y}, \tilde{X}, \tilde{Y})$, $\tilde{p} \in G_2(V(M))$, for any orthonormal (with respect to $\tilde{g}_u$) linear basis $\{\tilde{X}, \tilde{Y}\}$ in $\tilde{p}$, $u = \tilde{\mathrm{p}}(\tilde{p})$. Next we define $r, s : GF_2(M) \to IR$, by $r(p) = b(\beta(p))$, $s(p) = b(\gamma(p))$, $p \in GF_2(M)$. The maps r, s are the *horizontal* (resp. *vertical*) *sectional curvatures* of the Finsler space (M, E), such as introduced in [5], provided that g is given by $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j E$. Indeed, let $\tilde{R}$ be the curvature 2-form of the Miron connection (1.9) - (1.10). Consider the tensor fields $\tilde{R}(X, Y, \tilde{Z}, \tilde{W}) = g(\tilde{R}(\tilde{Z}, \tilde{W}) Y, X)$, and $R(X, Y, Z, W) = \tilde{R}(X, Y, \beta Z, \beta W)$, $S(X, Y, Z, W) = \tilde{R}(X, Y, \gamma Z, \gamma W)$. Then the following identities hold :

$$\begin{aligned} \tilde{B}(\tilde{X}, \tilde{Y}) \tilde{Z} &= \beta \tilde{R}(\tilde{X}, \tilde{Y}) L \tilde{Z} + \gamma \tilde{R}(\tilde{X}, \tilde{Y}) G \tilde{Z} \\ \tilde{B}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) &= \tilde{R}(L\tilde{X}, L\tilde{Y}, \tilde{Z}, \tilde{W}) + \tilde{R}(G\tilde{X}, G\tilde{Y}, \tilde{Z}, \tilde{W}) \end{aligned} \tag{3.2}$$

and consequently $r(p) = R(X, Y, X, Y)$, $s(p) = S(X, Y, X, Y)$, for $p \in GF_2(M)$ and for any orthonormal linear basis $\{X, Y\}$ in p.

4. MAIN RESULT

Let $p_0 \in GF_2(M)$, $u_0 = \mathrm{p}(p_0)$, be fixed. Let $\{X, Y\}$ be an orthonormal basis in p_0. Consider the curve $W: [0, 2\pi] \to p_0$ defined by $W(\theta) = (\cos\theta) X + (\sin\theta) Y$, $0 \leqq \theta \leqq 2\pi$. For simplicity we set $p_0^h = \beta(p_0)$, $p_0^v = \gamma(p_0)$; therefore $\theta \to \beta(W(\theta))$ (resp. $\theta \to \gamma(W(\theta))$) is a curve in p_0^h (resp. in p_0^v). With standard arguments) there exists a number $r > 0$ such that :

$$\begin{aligned} & t \beta W(\theta) \in W_{\tilde{0}} \cap N_{u_0} \\ & t \gamma W(\theta) \in W_{\tilde{0}} \cap \operatorname{Ker}(d_{u_0} \pi) \end{aligned} \tag{4.1}$$

for any $0 \leqq t \leqq r$. Therefore, the following curves are well defined, i.e. C_θ^h, $C_0^v : [0, r] \to V(M)$ given by :

$$G_\theta^h(t) = \exp_{u_0} t \beta W(\theta), \quad C_\theta^v(t) = \exp_{u_0} t \gamma W(\theta) \tag{4.2}$$

for any $0 \leqq \theta \leqq 2\pi$, $0 \leqq t \leqq r$. Moreover we consider the curves C^h, $C^v : [0, 2\pi] \to V(M)$ given by :

$$C^h(\theta) = C_0^h(r), \ C^v(\theta) = C_0^v(r). \tag{4.3}$$

Let $L(C^v)$, $L(C^h)$ be respectively given by

$$L(C^v) = \int_0^{2\pi} \tilde{g}_{C^v(\theta)} \left(\frac{dC^v}{d\theta}(\theta), \frac{dC^h}{d\theta}(0) \right) d\theta ,$$

$$L(C^h) = \int_0^{2\pi} \tilde{g}_{C^h(\theta)} \left(\frac{dC^h}{d\theta}(\theta), \frac{dC^h}{d\theta}(\theta) \right) d\theta.$$

We may formulate the following :

Theorem 4.1. *Let (M, g) be a generalized Finsler space carrying the non-linear connection N. Let $s : GF_2(M) \to IR$ be the vertical sectional curvature associated with the Miron connection determined by the pair (g, N). Then :*

$$s(p_0) = \lim_{\substack{r \to 0 \\ r \neq 0}} \frac{3}{\pi r^3} \{L(C^v) - 2\pi r\} \tag{4.4}$$

for each $p_0 \in GF_2(M)$, where C^v is given by (4.2).

It is an open problem to establish a geometrical interpretation similar to (4.4) for the horizontal sectional curvature r of (M, g, N).

5. JACOBI FIELDS ON GENERALIZED FINSLER SPACES

Let us put $\alpha^v(\theta, t) = C^v_\theta(t)$, $0 \leqq \theta \leqq 2\pi$, $0 \leqq t \leqq r$, with the notations in § 4. By (4.2) it follows that the family $\{C^v_\theta\}_{0 \leqq \theta \leqq 2\pi}$ consists of autoparallel curves of $\tilde{\nabla}$ with the initial data $(u_0, \gamma W(\theta))$. Clearly α^v is a variation of C^v_0, in the sense of [8, p.63], vol.II. Let then J^v_t be the infinitesimal variation induced by the variation α^v. We need to recall that J^v is a vector field along the 2-parameter surface α^v in $V(M)$ given by :

$$\begin{aligned} & J^v(\alpha^v(\theta, t)) = J^v_\theta(t) \\ & J^v_\theta(t) = \frac{\partial \alpha^v}{\partial \theta}(\theta, t) \\ & \frac{\partial \alpha^v}{\partial \theta}(\theta, t) = (d_\theta \alpha_t) \frac{d}{d\theta}\bigg|_\theta \\ & \alpha^v_t(\theta) = \alpha^v(\theta, t). \end{aligned} \tag{5.1}$$

Note that :

$$J^v_\theta(0) = 0 \quad , \quad 0 \leqq \theta \leqq 2\pi. \tag{5.2}$$

Let $u_0 \in V(M)$ be fixed. Put for brevity $W^v_{\tilde{0}} = W_{\tilde{0}} \cap \mathrm{Ker}(d_{u_0}\pi)$. Consider $\tilde{X}_0 \in W^v_{\tilde{0}}$ and the curve γ_0 in $V(M)$ defined by :

$$\gamma_0(t) = \exp_{u_0} t \tilde{X}_0 \tag{5.3}$$

for small values of the parameter t. Next we consider the first order ordinary differential system :

$$\tilde{\nabla}_{\frac{d\sigma}{dt}} \tilde{Z} = 0 \tag{5.4}$$

where $\sigma : [0, 1] \to V(M)$ is a given differentiable curve in $V(M)$. Let then $T^*_{\sigma,t} : T_{\sigma(0)}(V(M)) \to T_{\sigma(t)}(V(M))$ be the parallel displacement operator along σ, associated with (5.4). That is, if $\tilde{Z}$ is the unique solution of (5.4) with initial data $\tilde{Z}(0) = \tilde{Z}_0$ then $T^*_{\sigma,t}(\tilde{Z}_0) = \tilde{Z}(t)$, for any $\tilde{Z}_0 \in T_{\sigma(0)}(V(M))$. We establish :

Lemma 5.1. *For an arbitrary smooth curve* $\sigma : [0, 1] \to V(M)$ *one has* :

$$P_2 \circ T^*_{\sigma,t} = T^*_{\sigma,t} \circ P_2 \tag{5.5}$$

for any $0 \leqq t \leqq 1$.

Proof. Let $\tilde{X} \in T_u(V(M))$ and $\tilde{Z}$ the unique solution of (5.4) with $\tilde{Z}(0) = P_2 \tilde{X}$. Then $0 = P_2 \tilde{\nabla}_{\frac{d\sigma}{dt}} \tilde{Z} = \tilde{\nabla}_{\frac{d\sigma}{dt}} P_2 \tilde{Z}$, by our (1.13), i.e. $P_2 \tilde{Z}$ is a solution of (5.4). Moreover $(P_2 \tilde{Z})(0) = P_2 P_2 \tilde{X} = \tilde{Z}(0)$. Consequently $P_2 \tilde{Z} = \tilde{Z}$, and (5.5) holds, Q.E.D.

Let us replace now σ in (5.4) by the curve (5.3). By the very definition of γ_0, its tangent gives a solution of (5.4) $\left(\text{since } \gamma_0 \text{ is an auto-parallel curve of the linear connection (1.11) and } \frac{d\gamma_0}{dt}(0) = \tilde{X}_0\right)$. Applying Lemma 5.1. one has :

$$\frac{d\gamma_0}{dt}(t) = T^*_{\gamma_0,t}(\tilde{X}_0) = T^*_{\gamma_0,t}(P_2 \tilde{X}_0) = P_2 T^*_{\gamma_0,t}(\tilde{X}_0) = P_2 \frac{d\gamma_0}{dt}(t).$$

It follows that (5.3) *is a vertical curve provided that* $\tilde{X}_0$ *is vertical. Thus* :

$$(d_{\gamma_0(t)} \pi) \frac{d\gamma_0}{dt}(t) = 0$$

or $\pi \circ \gamma_0 = \text{constant}$, i.e. the curve (5.3) lies entirely in the fibre $V_{x_0} = \pi^{-1}(x_0) \subset V(M)$, $x_0 = \pi(u_0)$. The result obtained in terms of the curve (5.3) might be equally applied to the curve C_0^ν given by (4.2). Therefore :

$$C_0^\nu(t) \in V_{x_0}, \quad 0 \leqq \theta \leqq 2\pi, \quad 0 \leqq t \leqq r.$$

In addition to (5.1) we consider :

$$\frac{\partial \alpha^\nu}{\partial t}(\theta, t) = (d_t \alpha_\theta^\nu) \left.\frac{d}{dt}\right|_t$$

$$\alpha_0^\nu(t) = \alpha^\nu(\theta, t).$$

By (1.14) one has :

$$\tilde{A}(\gamma X, \gamma Y) = \gamma S^1(X, Y) = 0 \tag{5.6}$$

for any $X, Y \in \Gamma(V(M), \pi^{-1} T(M))$. Let us define :

$$\frac{D\tilde{X}}{\partial t}(\theta, t) = (\tilde{\nabla}_{\frac{\partial \alpha^v}{\partial t}} \tilde{X})_{\alpha^v(\theta,t)} \tag{5.7}$$

for any tangent vector field $\tilde{X}$ of $V(M)$ defined along the 2-parameter surface α^v in $V(M)$. Since C^v_θ lies entirely in V_{x_0}, and V_{x_0} is the maximal integral manifold of the vertical distribution Ker $(d\pi)$ passing through u_0, one obtains :

$$\frac{\partial \alpha^v}{\partial t}(\theta, t), \ \frac{\partial \alpha^v}{\partial \theta}(\theta, t) \in \text{Ker}\,(d_{\alpha^v(\theta,t)}\pi) \tag{5.8}$$

for $0 \leqq \theta \leqq 2\pi$, $0 \leqq t \leqq r$. Using (5.6) - (5.8) we derive :

$$\frac{D J^v}{\partial t}(0, 0) = \left\{ \tilde{\nabla}_{\frac{\partial \alpha^v}{\partial \theta}} \frac{\partial \alpha^v}{\partial t} \right\}_{u_0} \tag{5.9}$$

since $\left[\frac{\partial \alpha^v}{\partial t}, \frac{\partial \alpha^v}{\partial \theta}\right] = \theta$.

6. PROOF OF THE MAIN RESULT

Let $\tilde{\pi} : T(V(M)) \to V(M)$ be the natural projection of the tangent bundle over $V(M)$. We consider the natural imbedding η_t: $T(V(M)) \to T(T(V(M)))$, $t \in IR$, defined as follows: Let $\tilde{X}_0 \in T(V(M))$. Consider the curve $a(t) = t\tilde{X}_0$ in $T(V(M))$. Set :

$$\eta_t(\tilde{X}_0) = \frac{da}{dt}(t). \tag{6.1}$$

Actually, if $\tilde{\pi}(\tilde{X}_0) = u_0$, $u_0 \in V(M)$, then $a(t)$ is a curve in $T_{u_0}(V(M))$. Therefore, its tangent vector at $a(t)$ is an element of $T_{t\tilde{X}_0}(T_{u_0}(V(M)) = \text{Ker}\,(d_{t\tilde{X}_0}\tilde{\pi})$, $t \in IR$. Let us consider now the curve (5.3) with $\tilde{X}_0 \in W_{\tilde{0}}$ not necessarily vertical. We may rewrite it :

$$\gamma_0(t) = \exp_{u_0} a(t) \tag{6.2}$$

for small enough values of t; taking the differential of (6.2) at t furnishes :

$$\frac{d\gamma_0}{dt}(t) = (d_{a(t)} \exp_{u_0})\, \eta_t(\tilde{X}_0). \tag{6.3}$$

Take (6.3) at $t = 0$; since γ_0 is an auto-parallel curve of (1.11) with initial data $(u_0, \tilde{X}_0)$ it follows :

$$(d_{u_0} \exp_{u_0})\, \eta_0 \tilde{X}_0 = \tilde{X}_0. \tag{6.4}$$

We apply the results given by (6.3) - (6.4) to the curve C^v_θ. Thus one has :

$$\frac{\partial \alpha^{\nu}}{\partial t}(\theta, 0) = \gamma\, W(\theta)\,, \quad 0 \leqq \theta \leqq 2\pi\,. \tag{6.5}$$

Let $(x^a) = (x^i, y^i)$, $1 \leqq a \leqq 2n$, be the natural local coordinates on $V(M)$. Let Γ^a_{bc} be the corresponding local coefficients of the linear connection (1.11). The right hand side of (5.9) is locally given by :

$$\left\{ \tilde{\nabla}_{\frac{\partial \alpha^\nu}{\partial \theta}} \frac{\partial \alpha^\nu}{\partial t} \right\}^a_{u_0} = \frac{\partial^2 \alpha^a}{\partial \theta\, \partial t}(0, 0) + \Gamma^a_{bc}(\alpha^\nu(0, 0))\, \frac{\partial \alpha^b}{\partial \theta}(\theta, 0)\frac{\partial \alpha^c}{\partial t}(\theta, 0) \tag{6.6}$$

where $\alpha^\nu(\theta, t) = (\alpha^1(\theta, t), \ldots, \alpha^{2n}(\theta, t))$. Let $W^1(\theta) = X^i \cos\theta + Y^i \sin 0$ be the components of the Finslerian vector field $W(\theta)$ on M. Our (6.5) leads to :

$$\frac{\partial \alpha^i}{\partial t}(\theta, 0) = \theta\,, \quad \frac{\partial \alpha^{n+i}}{\partial t}(\theta, 0) = W^i(\theta) \tag{6.7}$$

for $1 \leqq i \leqq n$. By (5.1) - (5.2) and (6.6) - (6.7) one has

$$\frac{D\, J^\nu}{\partial t}(\theta, 0) = \frac{d\, W^1}{d\theta}(\theta)\, \dot{\partial}_i \Big|_{u_0}$$

or :

$$\frac{D\, J^\nu}{\partial t}(\theta, 0) = \gamma\, W\left(\theta + \frac{\pi}{2}\right). \tag{6.8}$$

For each $\tilde{X} \in T_u(V(M))$ we put $\|\tilde{X}\| = \tilde{g}_u(\tilde{X}, \tilde{X})^{1/2}$. We consider the function $f^\nu_\theta : [0, r] \to (0, +\infty)$ given by :

$$f^\nu_\theta(t) = \|J^\nu_\theta(t)\|^2 \quad, \quad 0 \leqq t \leqq r\,. \tag{6.9}$$

We develop (6.9) as a Taylor series :

$$f^\nu_\theta(t) = \sum_{k=0}^{4} \frac{t^k}{k!}(D^k f^\nu_\theta)(0) + o(t^5) \tag{6.10}$$

and compute $D^k f^\nu_\theta$, where $D^k = \frac{\partial^k}{\partial t^k}$, $0 \leqq k \leqq 4$. By (5.2), (6.8) one obtains:

$$\begin{aligned} &f^\nu_\theta(0) = 0 \\ &(D f^\nu_\theta)(0) = 0 \\ &(D^2 f^\nu_\theta)(0) = 0 \end{aligned} \tag{6.11}$$

since the connection (1.11) verifies (1.12). How (5.1) is the infinitesimal variation induced by the variation α^ν; by Theorem 1.2. in [8, p.64] one obtains :

$$\widetilde{\nabla}^2_{\frac{\partial \alpha^\nu}{\partial t}} J^\nu + \widetilde{\nabla}_{\frac{\partial \alpha^\nu}{\partial t}} \widetilde{A}\left(J^\nu, \frac{\partial \alpha^\nu}{\partial t}\right) + \widetilde{B}\left(J^\nu, \frac{\partial \alpha^\nu}{\partial t}\right)\frac{\partial \alpha^\nu}{\partial t} = 0 . \qquad (6.12)$$

Take (6.12) at u_0. By (5.1), (5.6), (5.8) it turns into:

$$\{\widetilde{\nabla}^2_{\frac{\partial \alpha^\nu}{\partial t}} J^\nu\}_{u_0} = 0 . \qquad (6.13)$$

Consequently:

$$(D^3 f^\nu_\theta)(0) = 0 . \qquad (6.14)$$

Let $S(X, Y)Z = \widetilde{R}(\gamma X, \gamma Y)Z$ be the vertical curvature of the Miron connection, $X, Y, Z \in \Gamma(V(M), \pi^{-1} T(M))$. By (3.2) one obtains $\widetilde{B}(\gamma X, \gamma Y)\gamma Z = \gamma S(X, Y)Z$. Using (1.12) we have:

$$(D^4 f^\nu_\theta)(0) = 8\, \widetilde{g}_{u_0}\left(\{\widetilde{\nabla}^3_{\frac{\partial \alpha^\nu}{\partial t}} J^\nu\}_{u_0}, \{\widetilde{\nabla}_{\frac{\partial \alpha^\nu}{\partial t}} J^\nu\}_{u_0}\right) . \qquad (6.15)$$

Take the covariant derivative of the Jacobi equation (6.12) in the direction $\frac{\partial \alpha^\nu}{\partial t}$. Moreover, take the inner product of the resulting equation by $\{\widetilde{\nabla}_{\frac{\partial \alpha^\nu}{\partial t}} J^\nu\}_{u_0}$. Then (6.15) becomes:

$$(D^4 f^\nu_\theta)(0) = 8\, \widetilde{g}_{u_0}\left(\widetilde{\nabla}_{\frac{\partial \alpha^\nu}{\partial t}} \widetilde{B}\left(J^\nu, \frac{\partial \alpha^\nu}{\partial t}\right)\frac{\partial \alpha^\nu}{\partial t}, \widetilde{\nabla}_{\frac{\partial \alpha^\nu}{\partial t}} J^\nu\right). \qquad (6.16)$$

On the other hand:

$$\widetilde{\nabla}_{\frac{\partial \alpha^\nu}{\partial t}} \widetilde{B}\left(J^\nu, \frac{\partial \alpha^\nu}{\partial t}\right)\frac{\partial \alpha^\nu}{\partial t} = \widetilde{B}\left(\widetilde{\nabla}_{\frac{\partial \alpha^\nu}{\partial t}} J^\nu, \frac{\partial \alpha^\nu}{\partial t}\right)\frac{\partial \alpha^\nu}{\partial t} . \qquad (6.17)$$

Now take (6.17) in u_0 and use (6.8). From the resulting equation let us substitute in (6.16). We obtain:

$$(D^4 f^\nu_\theta (0) = -8\, \widetilde{g}_{u_0}\left(\widetilde{B}\left(\gamma W\left(\theta + \frac{\pi}{2}\right), \gamma W(\theta)\right)\gamma W(\theta), \gamma W\left(0 + \frac{\pi}{2}\right)\right) . \qquad (6.18)$$

Moreover, in terms of the vertical curvature tensor:

$$(D^4 f^\nu_\theta)(0) = -8\, S_{u_0}\left(W\left(\theta + \frac{\pi}{2}\right), W(\theta), W\left(\theta + \frac{\pi}{2}\right), W(\theta)\right) . \qquad (6.19)$$

At this point we may substitute in (6.10) from the formulae (6.11), (6.14) and (6.19). This procedure gives:

$$f^\nu_\theta (t) = t^2 \left\{1 - \frac{t^2}{3} S_{u_0}\left(W\left(\theta + \frac{\pi}{2}\right), W(\theta), W\left(\theta + \frac{\pi}{2}\right), W(\theta)\right) + o(t^2)\right\} . \qquad (6.20)$$

As $(1 - \delta)^{1/2} = 1 - \frac{1}{2}\delta + o(\delta^2)$ we obtain:

$$L(C^r) = 2\pi r + \int_0^{2\pi} S_{u_0}\left(W\left(\theta + \frac{\pi}{2}\right), W(\theta), W\left(\theta + \frac{\pi}{2}\right), W(\theta)\right) d\theta + o(r^3). \quad (6.21)$$

Now $\left\{W(\theta), W\left(\theta + \frac{\pi}{2}\right)\right\}$ is an orthonormal basis in $p_0 \in GF_2(M), u_0 = \rho(p_0)$, and thus (6.21) leads to (4.4), Q.E.D.

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ÖZET

Bu çalışmada, genelleştirilmiş bir M Finsler uzayı verildiğine göre, M üzerindeki bütün teğet doğrultularının $V(M) = T(M) - 0$ manifoldunun yapısı incelenmektedir.