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ON CONTINUITY OF THE SAMPLE FUNCTION OF GAUSSIAN RANDOM VECTOR FIELD ON THE HILBERT SPACE

AL-MADANI M. GHALEB

Lipschitz conditions of order α corresponding to Gaussian Random vector field on a Hilbert space are found, also some conditions for continuity with probability one is determined. The necessary and sufficient condition for a Random vector field to be isotropic is also investigated.

1. Introduction

Conditions for absolute continuity corresponding to the sample function of a Gaussian Random vector field have not yet been sufficiently investigated.

In this paper an attempt has been made to find Lipschitz conditions of order α corresponding to Gaussian Random vector field on a Hilbert space. In addition to this some conditions for continuity with one probability has also been investigated.

Let H be perfect separable Hilbert space. Later under H we shall comprehend the space

$$H=\left\{t:(t_1,\ldots,t_n,\ldots),\ \sum_{n=1}^{\infty} t_n^2 < \infty\right\}.$$

T is a compact subset in H such that

$$T=\left\{t: a_n \le t_n \le a_n + \frac{1}{2^n}\ ,\ (a_1,\ldots,a_n,\ldots)\in H\right\}.$$

Let also (Ω, β, P) denote probability space (see for example {1}), $X(t, \omega)$ is a column vector, X' is the corresponding row vector and G denote a rotation group in H, i.e. collection of all one to one and measurable transformations from H into H.

Definitions. 1. A function $X(t,\omega)$ on $H\times\Omega$ is called a random vector field if for any fixed element t, $X(t,\omega)$ is a measuring function with respect to the σ algebra β, for any fixed ω the Random vector field $X(t,\omega)$ is called a sample function of the Random vector field.

2. The Random vector field $X(t,\omega)$ is called isotropic if for any t and S from H and $\forall\ g\varepsilon G$

$$E\{X(t,\omega).\,X'(s,\omega)\} = E\{X(gt,\omega)\,.\,X'(gs,\omega)\}.$$

3. The Random vector field $X(t,\omega)$ is continuous in mean square if

$$\lim_{t\to s}\sum_{i=1}^{n}|(X_i(t,\omega) - X_i(s,\omega)|^2 = 0.$$

Consider that $X(t,\omega)$ is an isotropic random n vector field with independent components, continuous in mean square with $E\,X(t,\omega) = 0$ and

$E\,[\{X(t,\omega)\}\,\{X'(s,\omega)\}] = I$,where I is the unit matrix, then the n by n matrix

$$E\,\{X(t,\omega)\,.\,X'(s,\omega)\} = B(t,s) = B(||\,t-s\,||)$$

will be positive definite continuous kernel on $H\times H$. According to Schoenberg [2] we may write

$$E\,\{X(t,\omega)\,.\,X'(s,\omega)\} = \int_0^\infty e^{-\lambda||t-s||^2}d\phi(\lambda), \tag{1}$$

where $\phi_{ij}(\lambda)$, $i,j = \overline{1,n}$ is bounded nondecreasing function.

2. Main Results

Theorem 1. If $X(t,\omega)$ is a Gaussian Random n vector field with independent components and $\forall\ t,\ s\in H$

$$E||\,X(t,\omega) - X(s,\omega)\,||^2 \leq \frac{||\,t-s\,||^{2\alpha}}{|\ln|\,|t-s|\,||}\,.\,C_1, \tag{2}$$

where C_1 is a positive constant, then $X(t,\omega)$ will satisfy on T with one probability Lipschitz conditions of order α.

Outlines of Proof. We observe that

$$E\,[\,\{X(t,\omega) - X(s,\omega)\}\,.\,\{\,X(t,\omega) - X(s,\omega)\}\,'\,] = 2\,[I - B(t,s)\,],$$

where $B(t,s)$ is a diagonal matrix given by the equation

$$B(t,s) = \int_0^{\infty} e^{-\lambda ||t-s||^2} \, d\phi(\lambda).$$

From the assumption in the Theorem 1, taking $C_1 = ||C||^2$, where C is a constant vector we can assert

$$E\,|X_i(t,\omega) - X_i(s,\omega)|^2 = 2\,(1 - B_{ii}(t,s)) \leq \frac{||t-s||^{2\alpha}}{|\ln||t-s|||} \,.\, C_i^2$$

$\forall i = \overline{1,n}$. An easy calculation shows that the previous inequality can be written as follows:

$$E\,|X_i(t,\omega) - X_i(s,\omega)|^2 \leq ||t-s||^{2\alpha} \,.\, C_i^2$$

$\forall i = \overline{1,n}$. Now by using Theorem 1, in {3} we see that $X_i(t,\omega)$ will satisfy on T with one probability Lipschitz conditions of order α, i.e.,

$$|X_i(t,\omega) - X_i(s,\omega)| \leq \delta_i \,.\, ||t-s||^{\alpha}$$

or

$$||X(t,\omega) - X(s,\omega)|| \leq ||\delta|| \,.\, ||t-s||^{\alpha} \,,$$

where δ is a constant vector.

This proves Theorem 1.

Theorem 2. If $X(t,\omega)$ is a Gaussian Random vector field with independent components on H and $\forall t,\ s \in H$

$$E\,||X(t,\omega) - X(s,\omega)||^2 \leq Q(||t-s||) \,.\, C_1 \,,$$

where C_1 is a positive constant,

$$Q(||t-s||) = \frac{1}{|\ln||t-s|||^{1+\delta}} \quad (\delta > 0) \tag{3}$$

or

$$\frac{1}{|\ln||t-s|||\,.\,|\ln|\ln||t-s||||^{2+\delta}} \quad (\delta > 0). \tag{4}$$

Then the sample function $X(t,\omega)$ is continuous on T with one probability.

Outlines of Proof. Since the components of the vector $X(x,\omega)$ are independent, we may write

$$E \mid \{X(t,\omega) - X(s,\omega)\} \, . \, \{X(t,\omega) - X(s,\omega)\}' \mid =$$

$$\begin{bmatrix} E \mid X_1(t,\omega) - X_1(s,\omega) \mid^2 & 0 \\ 0 & E \mid X_n(t,\omega) - X_n(s,\omega) \mid^2 \end{bmatrix}.$$

If (3) or (4) holds then we assert that

$$E \mid X_i(t,\omega) - X_i(s,\omega) \mid^2 \leq \frac{1}{\mid \ln \mid \mid t - s \mid \mid \mid^{4+\delta}} C_i^2$$

$\forall i = \overline{1,n}$. Using theorem (2) in [3] we can find a constant $\delta_i : i = \overline{1,n}$ such that the inequalities

$$\mid X_i(t,\omega) - X_i(s,\omega) \mid \leq \delta_i \mid\mid t - s \mid\mid$$

hold with one probability, i.e.,

$$\mid\mid X(t,\omega) - X(s,\omega) \mid\mid \leq \mid\mid \delta \mid\mid \, . \, \mid\mid t - s \mid\mid$$

with one probability, where δ is a constant vector.

Let L_n be the set of all possible $2n$-dimensional vectors $= (m_1, \ldots, m_n, k_1, \ldots, k_r)$ with natural components $(m_1 + m_2 + \ldots + m_n = n)$ and let us consider that

$$\gamma_n(t,\lambda,n) = e^{-\lambda(t,t)} (2\lambda)^{n/2} \frac{t_{m_1}^{k_1} \, . \, t_{m_2}^{k_2} \ldots . \, t_{m_n}^{k_n}}{\sqrt{k_1 ! \, k_2 ! \ldots k_n !}} \, . \tag{5}$$

Then the following theorems hold:

Theorem 3. The continuous in mean square Random vector field $X(t,\omega)$ is isotropic if and only if

$$X(t,\omega) = \sum_{n=0}^{\infty} \sum_{v \in l_n} \int_0^{\infty} \gamma_n(t,\lambda,n) \, . \, d\, Z_v^n(\lambda,\omega), \tag{6}$$

where $Z_v^n(s,\omega)$ is a sequence Random n vector measures on $(0,\infty)$, such that

$$E\,\{Z_v^n(s,\omega)\} = 0$$

$$E\,[\,Z_v^n(s_1,\omega)\,\} \, . \, \{Z_v^{n'}(s_2,\omega)\}'\,] = \delta_n^{n'} \, . \, \delta_v^{v'} \, \phi\,(s_1 \Lambda s_2). \tag{7}$$

Outlines of Proof. Consider that $X(t,\omega)$ is isotropic, then we write

$$B\,(r_{ts}) = \int_0^{\infty} e^{-\lambda \mid\mid t - s \mid\mid^2} \, d\,\phi(\lambda)\,.$$

Using {12.9, 5} in {4} we can write

$$e^{2\lambda(t,s)} = \sum_{n=0}^{\infty} (2\lambda)^n \frac{(t_{i_1} S_{i_1})^{k_1} \ldots\ldots (t_{i_n} S_{i_n})^{k_n}}{k_1 ! \ldots\ldots\ldots k_n !}$$

or

$$e^{2\lambda(t,s)} = \left[\sum_{n=0}^{\infty} \sum_{\nu \varepsilon I_n} (2\lambda)^{n/2} \frac{t_{i_1}^{k_1} \ldots t_{i_n}^{k_n}}{\sqrt{k_1 ! \ldots\ldots k_n !}} \right] .$$

$$\left[\sum_{n=0}^{\infty} \sum_{\nu \varepsilon I_n} (2\lambda)^{n/2} \frac{S_{i_1}^{k_1} \ldots\ldots S_{i_{n'}}^{k_{n'}}}{\sqrt{k_1 ! \ldots k_{n'} !}} \right] .$$

By substitution we get

$$B(r_{ts}) = E \iint_0^{\infty} e^{-\lambda(t,t)} . e^{-\lambda(s,s)} . \sum_{n=O}^{\infty} \sum_{\nu \varepsilon I_n} (2\lambda)^{n/2} - dZ_\nu^n (\lambda,\omega).$$

$$. \sum_{n'=O}^{\infty} \sum_{\nu' \varepsilon I_{n'}} (2\lambda)^{n/2} - d\, Z_{\nu'}^{n'} (\lambda,\omega)$$

or

$$X(t,\omega) = \int_0^{\infty} e^{-\lambda(t,t)} \sum_{n=0}^{\infty} \sum_{\nu \varepsilon I_n} (2\lambda)^{n/2} \frac{t_{i_1}^{k_1} \ldots\ldots t_{i_n}^{k_n}}{\sqrt{k_1 ! \ldots k_n !}} . d\, Z_\nu^n (\lambda,\omega).$$

Now if $X(t,\omega)$ satisfies the previous equation then it is easy to prove that

$$B(r_{ts}) = \int_0^{\infty} e^{-\lambda || t-s ||^2} \, d\,\phi(\lambda) .$$

The last equation means that $X(t, \omega)$ is isotropic.

Theorem 4. If

$$Tr \int_0^{\infty} \lambda \, d\,\phi(\lambda) < +\infty, \tag{8}$$

then the sample function $X(t,\omega)$ will satisfy on T with one probability Lipschitz conditions of order $\alpha < 1$.

Theorem 5. If

$$Tr\left\{\int_0^\infty \ln^{1+\delta}(1+\lambda)\, d\phi(\lambda)\right\} < \infty \tag{9}$$

for some $\delta > 0$, then the sample function $X(t, \omega)$ is continuous on T with one probability.

The proof of theorems 4 and 5 may be easily obtained.

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MATHEMATICS DEPARTMENT
COLLEGE OF ENGINEERING P.O.B. 9027
KING ABDULAZIZ UNIVERSITY
JEDDAH, SAUDI ARABIA

ÖZET

Bu çalışmada, bir Hilbert uzayı üzerindeki Gauss tesadüfi vektör alanına karşılık gelen α mertebeli Lipschitz koşulları bulunmakta ve olasılığı **1** olan bazı süreklilik koşulları belirlenmektedir. Aynı zamanda, bir tesadüfi vektör alanının izotropik olabilmesi için gerek ve yeter koşul araştırılmaktadır.