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CONTOUR INTEGRALS INVOLVING FOX'S H-FUNCTION AND THE MULTIVARIABLE H-FUNCTION

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The author presents two contour integrals for the multivariable H-function. Each of these contour integral formulae involves a product of Fox's H-function and the multivariable H-function. On account of the general nature of Fox's H-function and the multivariable H-function a large number of known as well as new and interesting contour integral formulae follow as special cases of these results.

1. Introduction

The multivariable H-function has been recently introduced by H.M. Srivastava and R. Panda is defined and represented in the following form

$$H(z_1 ,\ldots, z_r) = H^{0, u : (M', N') ; \ldots ; (M^{(r)}, N^{(r)})}_{v, w : (P', Q') ; \ldots ; (p^{(r)}, Q^{(r)})} \tag{1.1}$$

$$\begin{pmatrix} [(a) : A', \ldots, A^{(r)}] : [(b') : B'] : \ldots ; [(b^{(r)}) : B^{(r)}] ; \\ [(c) : C', \ldots, C^{(r)}] : [(d') : D'] ; \ldots ; [(d^{(r)}) : D^{(r)}] ; \end{pmatrix} z_1, \ldots, z_r \Bigg)$$

$$= (2\pi\omega)^{-r} \int_{£_1} \ldots \int_{£_r} U_1(s_1) \ldots U_r(s_r)\, V(s_1 ,\ldots, s_r)\, z_1^{s_1} \ldots z_r^{s_r}\, ds_1 \ldots ds_r ,$$

$$\omega = \sqrt{-1}$$

where

$$U_k(s_k) = \prod_{j=k}^{M^{(k)}} \Gamma(d_j^{(k)} - D_j^{(k)} s_k) \prod_{j=k}^{N^{(k)}} \Gamma(1 - b_j(k) + B_j^{(k)} s_k) \times \tag{1.2}$$

$$\times \Big\{ \prod_{j=1+M^{(k)}}^{Q^{(k)}} \Gamma(1 - d_j^{(k)} + D_j^{(k)} s_k) \prod_{j=1+N^{(k)}}^{P^{(k)}} \Gamma(b_j^{(k)} - B_j^{(k)} s_k) \Big\}^{-1}$$

$$V(s_1 ,\ldots, s_r) = \prod_{j=1}^{u} \Gamma\Big(1 - a_j + \sum_{k=1}^{r} A_j^{(k)} s_k\Big) \times \tag{1.3}$$

$$\times \Big\{ \prod_{j=u+1}^{v} \Gamma\Big(a_j - \sum_{k=1}^{r} A_j^{(k)} s_k\Big) \prod_{j=1}^{w} \Gamma\Big(1 - c_j + \sum_{k=1}^{r} C_j^{(k)} s_k\Big) \Big\}\cdot$$

The parameters

$$\begin{cases} a_j, j=1,\dots, v; & B_j^{(k)}, j=1,\dots,P^{(k)} \\ c_j, j=1,\dots, w; & d_j^{(k)}, j=1,\dots,Q^{(k)}; \forall k \in \{1,\dots,r\}, \end{cases} \tag{1.4}$$

are complex numbers, and the associated coefficients

$$\begin{cases} A_j^{(k)}, j=1,\dots, v; & B_j^{(k)}, j=1,\dots,P^{(k)} \\ C_j^{(k)}, j=1,\dots, w; & D_j^{(k)}, j=1,\dots,Q^{(k)}; \forall k \in \{1,\dots,r\}, \end{cases} \tag{1.5}$$

are positive real numbers such that

$$\Lambda_k = \sum_{j=1}^{u} A_j^{(k)} + \sum_{j=1}^{P^{(k)}} B_j^{(k)} - \sum_{j=1}^{w} C_j^{(k)} - \sum_{j=1}^{Q^{(k)}} D_j^{(k)} < 0 \tag{1.6}$$

and

$$\Omega_k = -\sum_{j=u+1}^{v} A_j^{(k)} + \sum_{j=1}^{N^{(k)}} B_j^{(k)} - \sum_{j=1+N^{(k)}}^{P^{(k)}} B_j^{(k)} - \sum_{j=1}^{w} C_j^{(k)} + \tag{1.7}$$

$$+ \sum_{j=1}^{M^{(k)}} D_j^{(k)} - \sum_{j=1+M^{(k)}}^{Q^{(k)}} D_j^{(k)} > 0, \forall k \in \{1,\dots,r\}$$

where the integers $u, M^{(k)}, N^{(k)}, v, P^{(k)}, w$ and $Q^{(k)}$ are constrained by the inequalities $0 \le u \le v$, $1 \le M^{(k)} \le Q^{(k)}, w \ge 0$, $0 \le N^{(k)} \le P^{(k)}, \forall k \in \{1,\dots,r\}$, and the equality in (1.6) holds for suitable restricted values of the complex variables $z_1, \dots, z_r$.

The contours $£_k$ are defined suitable and all the poles of the integrand are assumed to be simple.

The multiple integral in (1.1) converges absolutely under the conditions (1.7), when

$$|\arg(z_k)| < T_k \pi/2, \quad \forall_k \in \{1,\dots,r\}, \tag{1.8}$$

the points $z_k = 0$, $k = 1,\dots, r$, and various exceptional parameter values, being tactically excluded.

The following result is required in our investigation:

The series representation of Fox's H-function ([1], [2])

$$H_{p,q}^{m,n}\left[y \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_q, F_q) \end{matrix}\right] = \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\, F_G} \phi(g_s)\, y^{g_s} \tag{1.9}$$

where

$$\phi(g_s) = \prod_{j=1, j \neq G}^{m} \Gamma(f_j - F_j g_s) \prod_{j=1}^{n} \Gamma(1 - s_j + E_j g_s) \times \tag{1.10}$$

$$\times \left\{ \prod_{j=m+1}^{q} \Gamma(1 - f_j + F_j g_s) \prod_{j=1}^{p} \Gamma(e_j - E_j g_s) \right\}^{-1}$$

and

$$g_s = (f_G + s)/F_G . \tag{1.11}$$

We remark in passing that, throughout the present work, we shall assume that the convergence (and existence) conditions corresponding appropriately to the ones detailed above are satisfied by the multivariable H-functions involved.

With a view to facilitating the derivation of our main contour integrals (2.1) and (2.2) in the next section, we give here an elementary contour integral contained in the following

Lemma. If $h > 0$, $c > -Re(z)$ and $Re(\rho) > 0$, then

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ht} (t+z)^{-\rho} H_{p,q}^{m,n} \left[y(t+z)^{\sigma} \middle| \begin{matrix} (e_p, E_p) \\ (f_q, F_q) \end{matrix} \right] dt \tag{1.12}$$

$$= (h)^{\rho-1} e^{-hz} \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s \phi(g_s) (yh)^{g_s}}{s! \, F_G \Gamma(\rho + \sigma g_s)} .$$

Proof. The assertion (1.12) of the above lemma follows at once by applying the definition (1.9) in conjunction with the following form of the well known Hankel's contour integral for the Gamma function

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ht} (t+z)^{-\rho} dt = \frac{h^{\rho-1} e^{-hz}}{\Gamma(\rho)}, \quad h > 0, \tag{1.13}$$

where, for convergence, $c > -Re(z)$ and $Re(\rho) > 0$.

2. The main contour integrals

Our main results of the paper are the contour integrals contained in the following theorems:

Theorem 1. With Λ_k and Ω_k defined by (1.6) and (1.7), respectively, let

$$\Lambda_k \leq 0 \quad \text{and} \quad |\arg(z_k)| < \Omega_k \pi/2, \forall k \in \{1, \ldots, r\}, \tag{2.1}$$

where each of the equalities holds for suitably restricted values of the complex variables $z_1, \ldots, z_r$. Also let the function $H_{p,q}^{m,n}[y]$ be defined by (1.9), and let $h > 0$ and $c > -Re(z)$.

Then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ht}(t+z)^{-\rho} H_{p,q}^{m,n}\left[y(t+z)^{-\sigma}\left|\begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right.\right] \times \tag{2.2}$$

$$\times H(z_1(t+z)^{-\rho_1},\ldots,z_r(t+z)^{-\rho_r})\,dt =$$

$$= h^{\rho-1} e^{-hz}\sum_{G=1}^{m}\sum_{s=0}^{\infty}\frac{(-\mathrm{i})^s}{s!\,F_G}\phi(g_s)(yh)^{g_s}\times$$

$$\times H^{0,u:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w+\mathrm{i}:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left([1-\rho-\sigma g_s:\rho_1,\ldots,\rho_r],\right.$$

$$\left.\begin{matrix}[(a):A',\ldots,A^{(r)}]:[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(c):C',\ldots,C^{(r)}]:[(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}\; z_1 h^{\rho_1},\ldots,z_r h^{\rho_r}\right),$$

provided that $\rho_j > 0$, $j = 1,\ldots, r$, $\sigma > 0$, $|\arg y| < T\pi/2$ with

$$T = \sum_{j=1}^{n} E_j - \sum_{j=n+1}^{p} E_j + \sum_{j=1}^{m} F_j - \sum_{j=m+1}^{p} F_j > 0 \tag{2.3}$$

and

$$Re(\rho) > -c\sum_{j=1}^{r}\rho_j . \tag{2.4}$$

Theorem 2. Under the hypothesis preceding the assertion (2.2) of Theorem 1,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ht}(t+z)^{-\rho} H_{p,q}^{m,n}\left[y(t+z)^{-\sigma}\left|\begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right.\right]\times \tag{2.5}$$

$$\times H(z_1(t+z)^{\rho_1},\ldots,z_r(t+z)^{\rho_r})\,dt =$$

$$= h^{\rho-1} e^{-hz}\sum_{G=1}^{m}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,F_G}\phi(g_s)(yh)^{g_s}\times$$

$$\times H^{0,u:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v+1,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}];[\rho+\sigma g_s:\rho_1,\ldots,\rho_r]\\ [(c):C',\ldots,C^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}\; z_1 h^{-\rho_1},\ldots,z_r h^{-\rho_r}\right),$$

provided that $\rho_j > 0$, $j = 1,\ldots, r$, $\sigma > 0$, $|\arg y| < T\pi/2$

and

$$Re\,(\mathrm{p}) > -c \sum_{j=1}^{r} \rho_j \tag{2.6}$$

Proofs of Theorems 1 and 2. The contour integral formula (2.2) contained in Theorem 1, can be established if we first replace the multivariable *H*-function in the integrand by its multiple contour integral (1.1), change the order of integration, evaluate the inner most integral by appealing to the assertion (1.12) of our lemma and the interpret the resulting multiple contour integral as an *H*-function of *r* complex variables.

Similar is the derivation of the contour integral (2.5) of Theorem 2.

3. Applications

Each of contour integral formulae given by Theorem 1 and 2 of the preceding section possesses manifold generality. On specializing the parameters the multivariable *H*-function may be transformed into *G*-functions, *E*-functions, *H*-functions, Lauricella's functions, Appell's functions, Kampé de Fériet functions, Hypergeometric functions, Legendre functions, Bessel functions and several other higher transcendental functions in one, two or more arguments. Therefore, the results established in this paper are of general nature and hence encompass several cases of interest.

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ÖZET

Bu çalışmada çok değişkenli *H*-fonksiyonları için iki çevre integrali sunulmaktadır.