ON POWER SERIES AND MAHLER'S *U*-NUMBERS

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In the present paper it is proved that the function values of certain power series are *S*-numbers or *T*-numbers for some algebraic arguments according Mahler's classifification for transcendental numbers.

1. Introduction

Let

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \tag{1}$$

be a series with non-zero rational coefficients $c_n = b_n / a_n$ (a_n, b_n integers and $a_n > 1$) and let c_n satisfy the following conditions

$$\liminf_{n \to \infty} \frac{\log a_{n+1}}{\log a_n} = \sigma > 1, \tag{2}$$

$$\limsup_{n \to \infty} \frac{\log b_n}{\log a_n} = \theta < 1. \tag{3}$$

Then the radius of convergence of (1) is infinity.

By using a theorem of LeVêque [4, Theorem 4-15, p.148] which is a generalization of Thue-Siegel-Roth Theorem it is possible to prove that for a non-zero algebraic number α of degree smaller than $\sigma(1 - \theta) / (2u)$ the number $f(\alpha)$ is transcendental, where

$$u = \limsup_{n \to \infty} \frac{\log l.c.m.(a_0, a_1, \ldots, a_n)}{\log a_n} \tag{4}$$

is a finite number with $1 \leq u \leq \sigma / (\sigma - 1)$ (Zeren [5]).

It is the purpose of the present paper to give a necessary and sufficient condition for $f(\alpha)$ to be a *U*-number according to Mahler's classification for transcendental numbers. We prove

Theorem. Let $f(x)$ be a series as in (1) such that (2) and (3) hold. Suppose that α is a non-zero algebraic number of degree $m < \sigma(1 - \theta) / (2u)$.

1°) If

$$\limsup_{n\to\infty} \frac{\log a_{n+1}}{\log a_n} < +\infty \tag{5}$$

then $f(\alpha)$ is not a U-number, i.e. it is either an S-number or a T-number.

2°) If

$$\limsup_{n\to\infty} \frac{\log a_{n+1}}{\log a_n} = +\infty \tag{6}$$

then $f(\alpha)$ is a U-number of degree $\leq m$.

For the proof of the first part of the theorem we use essentially the following theorem of Baker [2, Theorem 1, p. 98]

Theorem (Baker). Suppose that ξ is a real or a complex number and $\varkappa > 2$. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of distinct numbers in an algebraic number field K with field heights $H_K(\alpha_1)$, $H_K(\alpha_2)$, ... such that

$$|\xi - \alpha_i| < (H_K(\alpha_i))^{-\varkappa} \tag{7}$$

and

$$\limsup_{i\to\infty} \frac{\log H_K(\alpha_{i+1})}{\log H_K(\alpha_i)} < +\infty \tag{8}$$

hold. Then ξ is either an S-number or a T-number.

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2. Lemmas

The following lemmas are used in the proof of the theorem.

Lemma 1. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ $(k \geq 1)$ be algebraic numbers which belong to an algebraic number field K of degree g, and let $F(y, x_1, \ldots, x_k)$ be a polynomial with rational integral coefficients and with degree at least one in y. If η is an algebraic number such that $F(\eta, \alpha_1, \ldots, \alpha_k) = 0$, then the degree of $\eta \leq dg$, and

$$H(\eta) \leq 3^{2dg+(d_1+\ldots+d_k)g} \,.\, H^g \,.\, (H(\alpha_1))^{d_1 g} \ldots (H(\alpha_k))^{d_k g} ,$$

where $H(\eta)$ is the height of η, H is the maximum of the absolute values of the coefficients of F, d_i $(i = 1, \ldots, k)$ is the degree of F in x_i $(i = 1, \ldots, k)$, d is the degree of F in y, and $H(\alpha_i)$ is the height of α_i $(i = 1, \ldots, k)$.

Proof. See İçen [1, p.25].

Lemma 2. Suppose that K is an algebraic number field of degree N and that ζ is an algebraic number in K with field height $H_K(\zeta)$. Let the field conjugates of ζ be $\zeta^{(1)} = \zeta, \zeta^{(2)}, \ldots, \zeta^{(N)}$ and let the coefficient of x^N in the field equation of ζ, with relatively prime integer coefficients, be h. Then

$$h \cdot \prod_{i=1}^{N} (1 + |\zeta^{(i)}|) < 6^N \cdot H_K(\zeta) .$$

Further, if $j_1, \ldots, j_s$ are s integers with $1 \leq j_1 < \ldots < j_s \leq N$ then

$$h \cdot \zeta^{(j_1)} \ldots \zeta^{(j_s)}$$

is an algebraic integer.

Proof. Se LeVêque [4, Theorem 4-2, pp. 124-125, and Theorem 2-21, pp. 63-65].

Lemma 3. Let ζ_1 and ζ_2 be different conjugates of an algebraic number of degree m and of height H. Then

$$|\zeta_1 - \zeta_2| \geq (4\,m)^{-(m-2)/2} \; ((m+1)\,H)^{-(2m-1)/2} .$$

Proof. See Güting [3, Theorem 8, p. 158].

In the remainder of this paper the inequalities hold for all sufficiently large indices and the real numbers $\varepsilon_1, \varepsilon_2, \ldots$ are positive and sufficiently small such that they are not depending on the varying indices.

Lemma 4. Let $f(x)$ be a series as in (1) such that (2), (3) and $1 < \sigma(1 - \theta)$ hold. Suppose that α is a non-zero algebraic number of degree m. Let $\beta_n = \sum_{\nu=0}^{n} c_\nu \alpha^\nu$ $(n = 0, 1, \ldots)$. Then the numbers of the consecutive terms of the sequence $\{\beta_n\}$ which are of degree $< m$ are bounded.

Proof. Let $K = \mathbf{Q}(\alpha)$, then $[K : \mathbf{Q}] = m$, $\beta_n \in K$. It follows from (2) that the sequence $\{a_n\}$ is monotonically increasing for all sufficiently large n and it holds

$$\lim_{n\to\infty} \frac{\log a_n}{n} = +\infty . \tag{9}$$

It follows further from (3)

$$|b_n| < a_n^{\theta+\varepsilon_1} < a_n \quad (0 < \varepsilon_1 < 1 - \theta) . \tag{10}$$

We assume that the assertion of the lemma is not true. Then there must exist a sequence $\{\Sigma_s\}$ such that

$\Sigma_s = \{\beta_{n_s+1}, \ldots, \beta_{n_s+q_s}\}$ $(n_s, q_s \to \infty$ as $s \to \infty)$ with $\deg \beta_{n_s} = m$ and $\deg \beta_\nu < m$ for $n_s + 1 \leq \nu \leq n_s + q_s$, where $\deg \beta$ denotes the degree of the algebraic number β.

Let $\beta_n^{(1)} = \beta_n$, $\beta_n^{(2)}$,..., $\beta_n^{(m)}$ be the field conjugates of β_n . For a pair (i,j) $(1 \leq i < j \leq m)$ the equations

$$\beta_\nu^{(i)} = \beta_\nu^{(j)} \quad (\nu = n, n+1, n+2)$$

can not be satisfied simultaneously. For otherwise we would get from

$$\frac{\beta_{n+2}^{(i)} - \beta_{n+1}^{(i)}}{\beta_{n+1}^{(i)} - \beta_n^{(i)}} = \frac{\beta_{n+2}^{(j)} - \beta_{n+1}^{(j)}}{\beta_{n+1}^{(j)} - \beta_n^{(j)}}$$

that $\alpha^{(i)} = \alpha^{(j)}$ which is a contradiction. It follows from here, $q_n \to \infty$ and the finiteness of the number of the pairs (i,j) that there exists an index pair (i,j) and a subsequence $\{\Sigma_s'\}$ of $\{\Sigma_s\}$ such that for every s it is possible to find terms $\beta_{n_t}, \beta_{n_{t+1}} \in \Sigma_s'$ with $n_{t+1} - n_t \geq 2$, $\beta_{n_t}^{(i)} = \beta_{n_t}^{(j)}$, $\beta_{n_{t+1}}^{(i)} = \beta_{n_{t+1}}^{(j)}$ and $\beta_\nu^{(i)} \neq \beta_\nu^{(j)}$ $(n_t < \nu < n_{t+1})$. Because of $\beta_{n_t+1}^{(i)} \neq \beta_{n_t+1}^{(j)}$ it follows that $(\alpha^{(i)})^{n_t+1} \neq (\alpha^{(j)})^{n_t+1}$. Furthermore we have

$$\beta_{n_t+1}^{(i)} - \beta_{n_t}^{(i)} = \beta_{n_t+1}^{(j)} - \beta_{n_t}^{(j)}$$

and hence

$$\sum_{\nu=1}^{n_{t+1}-n_t} c_{n_t+\nu} \left((\alpha^{(i)})^{n_t+\nu} - (\alpha^{(j)})^{n_t+\nu} \right) = 0 . \tag{11}$$

It follows from (2) and (10)

$$\left| \frac{c_{n+1}}{c_n} \right| \leq a_n^{-(1-\theta-\varepsilon_1)(\sigma-\varepsilon_2)+1} \quad (0 < \varepsilon_1 < 1 - \theta , 1 < \sigma - \varepsilon_2) . \tag{12}$$

From $1 < \sigma(1-\theta)$ we get $(1-\theta-\varepsilon_1)(\sigma-\varepsilon_2) - 1 > 0$. By (11) and (12) we obtain

$$|(\alpha^{(i)})^{n_t+1} - (\alpha^{(j)})^{n_t+1}| < 2(n_{t+1} - n_t - 1) . \max(1, |\alpha|)^{n_{t+1}} \left| \frac{c_{n_t+2}}{c_{n_t+1}} \right| . \tag{13}$$

It holds $H(\alpha^{n_t+1}) \leq \gamma_1^{n_t+1}$ and from Lemma 3

$$|(\alpha^{(i)})^{n_t+1} - (\alpha^{(j)})^{n_t+1}| \geq \gamma_2 \cdot \gamma_3^{-(n_t+1)} , \tag{14}$$

where the real constants $\gamma_1, \gamma_2, \gamma_3$ are positive and are not depending on n_t .

If $n_{t+1} - n_t$ is bounded for $t \to \infty$ then there exists a real constant $B > 0$ with $n_{t+1} - n_t - 1 \leq B$. Hence it follows from (12), (13) and (14)

$$a_{n_t+1}^{(1-\theta-\varepsilon_1)(\sigma-\varepsilon_2)-1} \leq \frac{2B}{\gamma_2} \max(1, |\alpha|)^B (\max(1, |\alpha|) \gamma_3)^{n_t+1}$$

which is a contradiction because of (9).

Hence $n_{t+1} - n_t$ is not bounded for $t \to \infty$. Therefore there exists an index pair $(p, r) \neq (i, j)$ and a subsequence $\{\Sigma_s''\}$ of $\{\Sigma_s'\}$ such that for every s it is possible to find terms $\beta_{n_u}, \beta_{n_{u+1}} \in \Sigma_s''$ with $n_{u+1} - n_u \geq 2, n_t < n_u < n_{u+1} < n_{t+1}$, $\beta_{n_u}^{(p)} = \beta_{n_u}^{(r)}, \beta_{n_{u+1}}^{(p)} = \beta_{n_{u+1}}^{(r)}$ and $\beta_\nu^{(p)} \neq \beta_\nu^{(r)}$ $(n_u < \nu < n_{u+1})$. We can show similarly that $n_{u+1} - n_u$ is not bounded for $u \to \infty$. If we go on, we get such terms in Σ_s for sufficiently large s with all different field conjugates because the number of the distinct pairs (i, j) is finite. This contradicts the definition of Σ_s. Hence the lemma is proved.

Lemma 5. Let $f(x)$, α and β_n be as in Lemma 4. If $\{\beta_{n_k}\}$ is the subsequence of the terms of degree m in $\{\beta_n\}$ then there is an integer k_0 so that it holds $\beta_{n_k} \neq \beta_{n_{k+1}}$ for all integers $k \geq k_0$.

Proof. If the assertion of the lemma were not true then it would hold $\beta_{n_k} = \beta_{n_{k+1}}$ for infinitely many k. Hence it would follow for infinitely many k

$$1 + \sum_{\nu=2}^{n_{k+1}-n_k} \frac{c_{n_k+\nu}}{c_{n_k+1}} \alpha^{\nu-1} = 0. \tag{15}$$

By Lemma 4 the number of the terms in (15) is bounded and by (12)

$$\lim_{k \to \infty} \frac{c_{n_k+\nu}}{c_{n_k+1}} = 0 \quad (\nu = 2, 3, \ldots, n_{k+1} - n_k).$$

Therefore we would get a contradiction from (15) and this proves Lemma 5.

3. Proof of First Part

We apply Lemma 1 on the polynomial

$$F(y, x) = A_n y - \sum_{\nu=0}^{n} A_n c_\nu x^\nu$$

where $A_n = l.\,c.\,m.\,(a_0, a_1, \ldots, a_n)$. Because of $F(\beta_n, \alpha) = 0$ we get from (4), (9) and (10)

$$H(\beta_n) \leq a_n^{u.m+\varepsilon_3}. \tag{16}$$

Let $\xi = f(\alpha)$. It follows from (2), (9), (10) and (16)

$$\begin{aligned} |\xi - \beta_n| &\leq a_{n+1}^{-(1-\theta-\varepsilon_4)} \quad (0 < \varepsilon_4 < 1 - \theta) \\ &\leq a_n^{-(1-\theta-\varepsilon_4)(\sigma-\varepsilon_2)} \\ &\leq H(\beta_n)^{-\varkappa} \end{aligned} \tag{17}$$

where $\varkappa = (1 - \theta - \varepsilon_4)(\sigma - \varepsilon_2) / (um + \varepsilon_3)$. Because of $m < \sigma(1-\theta)/(2u)$ we obtain $\varkappa > 2$ and $1 < \sigma(1-\theta)$.

We consider now the sequence $\{\beta_{n_k}\}$ $(k \geq k_0)$ in Lemma 5. We have for the terms of this sequence

$$H(\beta_{n_k}) = H_K(\beta_{n_k}) \,. \tag{18}$$

Let t_{n_k} be the coefficient of x^m in the field equation of β_{n_k} with relatively prime integer coefficients. We put

$$\wedge = t_{n_{k+1}} \,.\, t_{n_k} \,.\, \text{Norm}\,(\beta_{n_{k+1}} - \beta_{n_k}) \tag{19}$$

where

$$\text{Norm}\,(\beta_{n_{k+1}} - \beta_{n_k}) = \prod_{i=1}^{m} (\beta^{(i)}_{n_{k+1}} - \beta^{(i)}_{n_k}) \,. \tag{2θ}$$

Since $\beta_{n_{k+1}} \neq \beta_{n_k}$ it follows from (19) that $\wedge$ is the sum of products of conjugates of $\beta_{n_{k+1}}$ and β_{n_k}, all multiplied by $t_{n_{k+1}} \,.\, t_{n_k}$. It follows from Lemma 2 that $\wedge$ is a rational integer and hence we obtain

$$|\wedge| \geq 1 \,. \tag{21}$$

We now find an upper bound for $|\wedge|$. Since

$$|\beta^{(i)}_{n_{k+1}} - \beta^{(i)}_{n_k}| \leq (1 + |\beta^{(i)}_{n_{k+1}}|)(1 + |\beta^{(i)}_{n_k}|)$$

and from Lemma 2 and (18) we obtain

$$t_{n_k} \,.\, \prod_{i=1}^{m} (1 + |\beta^{(i)}_{n_k}|) \leq 6^m \,.\, H(\beta_{n_k}) \,. \tag{22}$$

It follows from (19) and (2θ) that

$$|\wedge| = |\beta_{n_{k+1}} - \beta_{n_k}| \,.\, \Big| t_{n_{k+1}} \,.\, t_{n_k} \,.\, \prod_{i=2}^{m} (\beta^{(i)}_{n_{k+1}} - \beta^{(i)}_{n_k}) \Big|$$

$$\leq |\beta_{n_{k+1}} - \beta_{n_k}| \,.\, 6^{2m} \,.\, H(\beta_{n_{k+1}}) \,.\, H(\beta_{n_k})$$

and

$$|\wedge| \leq |\beta_{n_{k+1}} - \beta_{n_k}| \,.\, 6^{2m} \,.\, (\max\{H(\beta_{n_{k+1}}), H(\beta_{n_k})\})^2 \,. \tag{23}$$

We obtain from (17) and $\sigma - \varepsilon_2 > 1$ that

$$|\beta_{n_{k+1}} - \beta_{n_k}| \leq |\xi - \beta_{n_{k+1}}| + |\xi - \beta_{n_k}|$$

$$\leq 2.\, a_{n_k}^{-(1-\theta-\varepsilon_4)}$$

and hence from (21) and (23)

$$a_{n_k}^{1-\theta-\varepsilon_4} \leq 2.6^{2m} \,.\, (\max \{ H(\beta_{n_{k+1}}) \,,\, H(\beta_{n_k}) \})^2 \,. \tag{24}$$

There are only a finite number of elements of K with bounded height and hence

$$\max \{ H(\beta_{n_{k+1}}) \,,\, H(\beta_{n_k}) \} \to \infty \text{ as } k \to \infty \,.$$

Thus from (24) on taking logarithmus it follows that

$$(1 - \theta - \varepsilon_4) \log a_{n_k} \leq (2 + \varepsilon_5) \max \{ \log H(\beta_{n_{k+1}}) \,,\, \log H(\beta_{n_k}) \} \,. \tag{25}$$

We define now inductively a sequence $\{k_i\}$. Let k_1 be the least positive integer for which the preceding inequalities and properties hold. Let i be a positive integer and we suppose that k_i has been defined and we take k_{i+1} as $k_i + 1$ or $k_i + 2$ according as $H(\beta_{n_{k_i+1}})$ is or is not greater than $H(\beta_{n_{k_i+2}})$. Then by definition

$$\max \{ \log H(\beta_{n_{k_{i-1}+2}}) \,,\, \log H(\beta_{n_{k_{i-1}+1}}) \} = \log H(\beta_{n_{k_i}}) \,. \tag{26}$$

By (5) there is a constant $c > 1$ such that

$$\log a_{n+1} < c \log a_n \,. \tag{27}$$

Hence from the definition of k_i it follows for all i

$$c^{-A} \,.\, \log a_{n_{k_i}} < \log a_{n_{k_{i-1}+1}} \,, \tag{28}$$

where A is an upper bound for $n_{k+1} - n_k$ by Lemma 4. From (25), (26) and (28) we obtain

$$(1 - \theta - \varepsilon_4) \,.\, c^{-A} \,.\, \log a_{n_{k_i}} < (2 + \varepsilon_5) \log H(\beta_{n_{k_i}}) \tag{29}$$

for all i. Hence we obtain from (16), (27) and (29)

$$\frac{\log H(\beta_{n_{k_i+1}})}{\log H(\beta_{n_{k_i}})} \leq \frac{(um + \varepsilon_3) \log a_{n_{k_i+1}}}{\frac{1 - \theta - \varepsilon_4}{c^A(2 + \varepsilon_5)} \log a_{n_{k_i}}} \leq \frac{(um + \varepsilon_3)\, c^A \log a_{n_{k_i}+1}}{\frac{1 - \theta - \varepsilon_4}{c^A(2 + \varepsilon_5)} \log a_{n_{k_i}}} \,. \tag{30}$$

We obtain from (5) and (30)

$$\limsup_{i \to \infty} \frac{\log H(\beta_{n_{k_i+1}})}{\log H(\beta_{n_{k_i}})} < \infty \,. \tag{31}$$

Finally we define a subsequence $\{\beta_{t_j}\}$ of $\{\beta_{n_{k_i}}\}$ so that we take $t_1 = 1$ and for each integer $j \geq 1$ we take t_{j+1} as the least integer in $\{n_{k_i}\}$ greater than t_j for which $H(\beta_{t_j})$ is less than $H(\beta_{t_{j+1}})$. It is possible to find such an index since the number of the algebraic numbers in K with bounded field height is finite and if in the sequence $\{\beta_n\}$ a term is repeated infinitely many times, then ξ must be in K because of the definition of β_n. Then we have

$$H(\beta_{t_1}) < H(\beta_{t_2}) < \ldots$$

and

$$\frac{\log H(\beta_{t_{j+1}})}{\log H(\beta_{t_j})} \leq \frac{\log H(\beta_{t_{j+1}})}{\log H(\beta_{t_{j+1}-1})}$$

hence

$$\limsup_{j\to\infty} \frac{\log H(\beta_{t_{j+1}})}{\log H(\beta_{t_j})} < \infty \,. \tag{32}$$

Moreover the terms of $\{\beta_{t_j}\}$ are all distinct because their heights are all distinct. We have further for the sequence $\{\beta_{t_j}\}$ from (17)

$$|\xi - \beta_{t_j}| < (H(\beta_{t_j}))^{-\varkappa} \tag{33}$$

with $\varkappa > 2$. We obtain from (32) and (33) the conditions (7) and (8) of Baker's Theorem for the sequence $\{\beta_{t_j}\}$ and hence the first part of the theorem is proved.

4. Proof of Second Part

Let $s_n = (\log a_{n+1})/(\log a_n)$. It follows from (6) that the sequence $\{s_n\}$ contains a subsequence $\{s_{nj}\}$ with $\lim\limits_{j\to\infty} s_{nj} = +\infty$. We consider now the sequence $\{\beta_{nj}\}$. No term in $\{\beta_{nj}\}$ can be repeated infinitely many times because of the transcendence of ξ. Hence there is a subsequence $\{\beta_{n_{j_q}}\}$ of $\{\beta_{nj}\}$ such that all its terms are different from each other and their heights increase monotonically. For this subsequence we get from (16) and (17)

$$|\xi - \beta_{nj_q}| \leq H(\beta_{nj_q})^{-\frac{1-\theta-\varepsilon_4}{um+\varepsilon_3}\cdot s_{nj_q}} \,. \tag{34}$$

Because of $\deg \beta_{nj_q} \leq m$ and $\lim\limits_{q\to\infty} s_{nj_q} = +\infty$ we get from (34) that ξ is a U^*-number of degree $\leq m$. From the equivalence of the Mahler's and Koksma's classification of transcendental numbers it follows that ξ is a U-number of degree $\leq m$.

REFERENCES

[1] İÇEN, O.Ş. : *Anhang zu den Arbeiten "Über die Funktionswerte der p-adischen elliptischen Funktionen I und II"*, İst. Üniv. Fen Fak. Mec. Serie A, **38** (1973), 25-35.

[2] BAKER, A. : *On Mahler's classification of transcendental numbers*, Acta Mathematica, 111 (1964), 97-120.

[3] GÜTING, R. : *Approximation of algebraic numbers by algebraic numbers*, Michigan Math. J., **8** (1961), 149-159.

[4] LEVÊQUE W.J. : Topics in Number Theory, Addison-Wesley, Massachusetts (1961), Vol. 2.

[5] ZEREN, B.M. : *Über die Transzendenz der Werte einiger schnell konvergenter Potenzreihen für algebraische Argumente*, Bulletin of the Technical University of İstanbul, **38** (1985), 473-496.

Ö Z E T

Bu çalışmada, bazı kuvvet serilerinin cebirsel argümanlar için aldığı değerlerin, Mahler'in transandant sayılar için verdiği tasnifteki S - veya T- sayıları oldukları ispat edilmektedir.