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A NOTE ON THE CHARACTERIZATION OF ET-SETS IN GLOBAL FIELDS

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In this note all extensions are assumed to be finite and k is a global field, i.e. an algebraic number field or an algebraic function field of one variable over a finite field $\mathbf{F}_q$ of constants. Let K/k be a Galois extension with group G and C be a conjugacy class in G. The set of unramified prime divisors P of k with the same Artin Symbol $\left(\frac{K/k}{P}\right) = C$ is called an *elementary Tchebotarev set* associated with K/k (abbreviated as ET-set) and denoted by $Art^{-1}_{K/k}\,(C)$. If K/k is abelian we call this set an *elementary abelian Tchebotarev set* (abbreviated as EAT-set). By Tchebotarev density law ([7], [8]) the set $Art^{-1}_{K/k}\,(C)$ has analytic density equal to $\frac{|C|}{|G|}$.

We are dealing with the following problem:

Problem. Given a set S of prime divisors of k, characterize the conditions under which there exists a Galois extension K/k such that S is (up to a finite set of primes) an ET-set associated with K/k, i.e. $S \doteq Art^{-1}_{K/k}\,(C)$ for some conjugacy class C in $G\,(K/k)$.

In this note we shall give an answer to this problem in a restricted form.

In [1] we prove the following result, which extends and improves a result of Gauthier [5, Théorème A]:

Theorem 1. Let K/k be a Galois extension of global fields, $G = G\,(K/k)$, G' the derived group of G, K' the fixed field of G' in K, C a conjugacy class of G. Then the following are equivalent :

(*i*) $|C| = |G'|$,

(*ii*) For every finite abelian extension N of k containing K'

$$Art^{-1}_{K/k}\,(C) \doteq \cup_{i=1}^{[N:K']} Art^{-1}_{N/k}\,(\psi_i)$$

where $\{\psi_1, \ldots, \psi'_{[N:K']}\} = \psi_1 G(N/K')$ is the coset of $G(N/k)$ determined by $\psi_1|_{K'} = C|_{K'}$,

(*iii*) There is an abelian extension N_1/k such that

$$Art_{K/k}^{-1}(C) \overset{\prime}{\supset} Art_{N_1/k}^{-1}(\psi)$$

for some $\psi \in G(N_1/k)$ ($\overset{\prime}{\supset}$ denotes the inclusion up to a finite set of primes of k).

Now we state the converse result which answers our initial problem in the following restricted form:

Theorem 2. Let k be a global field and S be a set of prime divisors of k which is (up to a finite set of prime divisors) a union of EAT-sets associated with some abelian extension N/k,

$$S \doteq \cup_{i=1}^{s} Art_{N/k}^{-1}(\psi_i)$$

such that $\{\psi_1, \ldots, \psi_s\} = \psi_1 H' \neq H'$ is a nontrivial coset modulo some subgroup H' of $G(N/k)$. Then there exist an infinite number of Galois extensions K/k such that the Galois group $G(K/k) = G$ has a conjugacy class C satisfying $|C| = |G'|$ and

$$S \doteq Art_{K/k}^{-1}(C).$$

Particular Case: $k = \mathbf{Q}$ or $k = \mathbf{F}_q(T)$.

For $k = \mathbf{Q}$, the field of rational numbers, an m—arithmetic progression is defined to be the set $\text{Prog}_m(r)$ of all prime numbers p which are congruent to $r \,(mod\ m)$, where $r \geq 1$ and $m \geq 2$ are relatively prime integers. Kronecker-Weber theorem states that every abelian extension of $\mathbf{Q}$ is contained in some m—th cyclotomic extension $\mathbf{Q}(\zeta_m)$.

For $k = \mathbf{F}_q(T)$, the rational function field, the role of $\mathbf{Q}(\zeta_m)$ is played by a type of extensions which we call (ν, n, M) — extensions in [2]. For $\nu \geq 0, n \geq 1$ integers and $M \in \mathbf{F}_q[T]$ a nonzero polynomial, the associated (ν, n, M)— extension N of $k = \mathbf{F}_q(T)$ is the composite $N = k_n \,.\, k(\Lambda_M) \,.\, L_\nu$ where k_n is the constant field extension of k of degree n, $k(\Lambda_M)$ and L_ν are some special extensions arising from Carlitz ([3], [4]) and Hayes [6] cyclotomic theory. The prime divisors of k which are ramified in N are those dividing $T'M$ and $P_\infty = \dfrac{1}{T}$. By the explicit construction of the maximal abelian extension of $\mathbf{F}_q(T)$ given by Hayes [6], we deduce an analogue of Kronecker-Weber Theorem : Every abelian extension K' of $k = \mathbf{F}_q(T)$ is contained in some (ν, n, M) — extension. In [2] we introduce the (ν, n, M)—arithmetic progressions as the sets $Prog_M^{\nu,n}(P) = Art_{N/k}^{-1}(\psi_P)$ where N is a (ν, n, M)—extension and $\psi_P = \left[\dfrac{N/k}{P}\right]$ is the Frobenius substitution of N/k at the prime P of $k = \mathbb{F}_q(T)$, identified with some prime polynomial P not dividing $T^{\nu}M$. These arithmetic progressions are completely characterized in [2].

Now using the Kronecker-Weber Theorem, respectively its p-analogue for function field case, it appears that every EAT-set is a union of arithmetic progressions associated with some N/k where N is an $m-$th cyclotomic extension, respectively a $(\nu, n, M)-$extension. So in case $k = \mathbf{Q}$ or $k = \mathbb{F}_q(T)$ it is enough to work with the arithmetic progressions which are more concrete sets than EAT-sets.

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