İstanbul Üniv. Fen Fak. Mat. Der. 49 (1990), 21-23

LIE ALGEBRAS WITH SOLVABLE WORD PROBLEM

N. EKİCİ

In this note we investigate associative division algebras having a presentation

$$A = < x, y \mid x^{m_\alpha} y^{m_\alpha} = y^{m_\alpha} x^{m_\alpha} \ , \ \alpha = 1, 2, \dots, k > , \ k \geq 1, \ m_\alpha \in \mathbf{Z}. \ (*)$$

The main results are the following :

Theorem A. Let A be an associative division algebra (*). Then A has a solvable word problem.

The theorem below reduces the problem to the case $k \leq 2$.

Theorem B. Let A be an associative division algebra presented by (*) and assume that the m_α are pairwise relatively prime. If $k \geq 3$ then A is abelian.

The abelian case is easy and we omit it.

THE NON-ABELIAN CASE, ŠIRŠOV'S THEOREM

In this section we will consider composition method. It is clear that, this is part of the theory of the computer algebra [3]. This method has several sources. The composition lemma is stated in explicit form for Lie algebras in the paper of Širšov [4]. It can be restated without difficulty to the associative case. We only have to remove the brackets in the Širšov argument. This was accomplished in explicit form by Bokut [2] and Bergman [1] in the language of associative algebras.

Let K be a field, X a set, $F = K < X >$ is a free associative algebra over the field K with the set of free generators X. We assume that the set of all words from X is linearly ordered and this order $\leq$ satisfies the minimality condition and is compatible with the word product. If w is a nonzero element of algebra F, by $l(w)$ we denote the length of w with respect to X. Leading word of w with nonzero coefficient will be denoted by $\overline{w}$.

Let us define the composition $(f, g)_w$ of the elements f, g of the algebra F with respect to the word w.

Definition. Let $w = \bar{f} a = b \bar{g}$ and the designated subwords $\bar{f}, \bar{g}$ of F are nonintersecting. Then the polynomial $(f, g)_w = \beta f a - \alpha b g$, where $f = \alpha \bar{f} + \ldots$, $g = \beta \bar{g} + \ldots$, $\alpha, \beta \in K$, is called the composition of the elements f, g with respect to the word w. Clearly either $(f, g)_w = 0$ or $\overline{(f, g)_w} < w$.

Definition. Let $S \subseteq F = K < X >$. We say that the set S is closed under composition if none of the leading words of the element $s \in S$ contains as a subword a leading word of another element from S and for any elements $f, g \in S$ and any composition $(f, g)_w$ in the algebra F we have the equality

$$(f, g)_w = \sum \alpha_i \, a_i \, s_i \, b_i \,,$$

where $s_i \in S$, $0 \neq \alpha_i \in K$, a_i, b_i are words and $\overline{a_i \, b_i \, s_i} < w$.

Composition Lemma. Let S be a closed subset of the algebra $F = K < X >$ closed with respect to the composition, then in the associative algebra $< X \mid s_i = 0, \; s_i \in S >$ all the words from X contain no subwords $\bar{s_i}$ $(s_i \in S)$ will serve as its base.

The case $k = 2$. Let

$$A = < x, y \mid x^m y^m = y^m x^m, \; x^n y^n = y^n x^n \,, \; m, n \in \mathbf{Z} >$$

and

$$S = \{ x^m y^m - y^m x^m \,, \; x^n y^n - y^n x^n \}.$$

Assume that $x < y$ and $m < n$.

Proposition. S is closed subset of the free associative algebra $F = K < \{x, y\} >$.

Proof. By calculating the composition of element of S we obtain the proposition.

Now we will consider the free algebra $F = K < \{x, y\} >$. Let S be as in the above proposition and J be the ideal generated by S. It is clear that $A \cong F/J$.

Theorem. The associative algebra

$$F/J \cong A = < x, y \mid x^m y^m = y^m x^m \,, \; x^n y^n = y^n x^n \,, \; m, \; n \in \mathbf{Z} >$$

has solvable word problem.

Proof. Let $u \in F$. If $\bar{u}$ doesn't contain $\bar{s}_i$ $(s_i \in S)$ as a subword then it is in the basis of A and it doesn't belong to J. If $\bar{u}$ contains $\bar{s}_i$ as a subword then consider the element $u_1 \in J$ such that $\bar{u} = \bar{u}_1$. Let $u = a_i \bar{s}_i b_i + w$ and $u_1 = a_i b_i s_i$ it is clear that $u_1 \in J$. Now consider the element $u - u_1$. $u - u_1$ has a smaller leading term than that of u. It belongs to the ideal J if and only if $u \in J$. Induction on the leading term completes the proof of the theorem.

Hence from this theorem and Theorem B we obtain the Theorem A.

Let us define the Lie product in the algebra A which has presentation (*) as $[a, b] = ab - ba$ for any $a, b \in A$. We obtain a Lie algebra L.

Corollary. L has solvable word problem.

REFERENCES

[1] BERGMAN, G. : *The diamond lemma for ring theory*, Adv. Math., **29** (1978), No. 2, 178-218.

[2] BOKUT, L.A. : *Embedding in simple associative algebras*, Algebra Logika, 15 (1976), No. 2, 117-142.

[3] BUCHBERGER, B. and LOOS, R. : Algebraic simplification in : Computer Algebra, Symbolic and Algebraic Computation, Springer, New York (1982).

[4] ŠIRŠOV, A.I. : *Some algorithmic problems for Lie algebras*, Sib. Math. Zh., 3 (1962), No. 2, 293-296.

UNIVERSİTY OF ÇUKUROVA
FACULTY OF SCIENCE AND ARTS
DEPARTMENT OF MATHEMATİCS
01330 BALCALI-ADANA