İstanbul Üniv. Fen Fak. Mat. Der. 49 (1990), 31-36

A NOTE ON BARELY TRANSITIVE PERMUTATION GROUPS SATISFYING MIN-2

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We recall that a group of permutations G of an infinite set Ω is called a barely transitive group if G acts transitively on Ω and every orbit of every proper subgroup is finite. An abstract group is called barely transitive, if it is isomorphic to some barely transitive permutation group. Recall also that [2] an infinite group G can be represented faithfully as a barely transitive permutation group if and only if G possesses a subgroup H such that $\bigcap_{x \in G} H^x = 1$ and $|K : K \cap H| < \infty$ for every proper subgroup $K < G$. The subgroup H is a point stabilizer of a barely transitive permutation group. Locally finite barely transitive groups are studied and the following theorem is proved in [5]:

Theorem [5] (1.2). *A locally finite barely transitive permutation group containing a nontrivial element of order p and satisfying min-p is isomorphic to* C_{p^∞}.

In the proof of the above theorem we invoke the classification of finite simple groups. In this paper we will prove the same result for the prime 2 without using the classification of finite simple groups and extend the above theorem by reducing the min-p condition on H.

By assuming some restrictions on point stabilizer H one might expect to obtain some results about the structure of a locally finite barely transitive group. On the lines of this idea we have three propositions which might be of interest. Proposition 4 might have independent interest.

Proposition 1. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If there exists a non-trivial element of order p in G and H satisfies min-p, then G satisfies min-p.*

Proof. Let Q be a Sylow p-subgroup of H. Then by [6] Q is a Černikov group. Since H is a proper subgroup of G the group $\mathbf{Q}$ is a proper subgroup hence residually finite. But a residually finite Černikov group is finite. Hence Q is finite. Let P be a Sylow p-subgroup of G. If G is a p-group, then finiteness of $|K : K \cap H|$ for each proper subgroup $K < G$ implies that each proper subgroup of G is finite hence G satisfies min-p.

Assume that P is a proper subgroup of G. Since $P \cap H$ is a p-subgroup Jo H it is contained in a Sylow p-subgroup of H which is finite. Barely transitivity implies $|P : P \cap H| < \infty$ hence P is finite i.e. G satisfies min-p.

Corollary. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If G contains a nontrivial element of order p and H satisfies min-p, then $G \cong C_{p^\infty}$.*

Proof. Use Proposition 1 and the above Theorem.

Theorem. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If G contains a nontrivial element of order 2 and H satisfies min-2, then $G \cong C_{2^\infty}$.*

Proof. By Proposition 1 G satisfies min-2. Let S be a Sylow 2-subgroup of G. Then S is *Černikov* [6] and so S has a divisible abelian normal subgroup of finite index. Residual finiteness of each proper subgroup of G [5] Lemma (2.13) and non residual finiteness of C_{2^∞} implies that either S is isomorphic to C_{2^∞} and so $G = S$ or S is proper and hence finite. In the first case we are done. We show that the second case is impossible.

Assume that G is a locally finite barely transitive group with finite Sylow 2-subgroups

a) each proper subgroup K of G satisfies $|K : O_{2'}(K)| < \infty$.

We prove this by induction on the order of Sylow 2-subgroups of proper subgroups of G.

Let $K < G$. If Sylow 2-subgroup of K is trivial group, then K is locally solvable by the Feit-Thompson theorem and $K = O_{2'}(K)$. Assume that in the set of proper subgroups of G if the order of Sylow 2-subgroup of K is less than the order of a Sylow 2-subgroup of G, then $|K : O_{2'}(K)| < \infty$. Let L be a proper subgroup of G containing Sylow 2-subgroup S of G. Let x be an involution in L. Since L is residually finite there exists a normal subgroup N_x of L such that $x \notin N_x$ and $|L : N_x|$ is finite. So order of Sylow 2-subgroup of N_x is less than the order of S. By the induction assumption $|N_x : O_{2'}(N_x)| < \infty$. As $O_{2'}(N_x)$ *char* $N_x \lhd L$ we have

$$|L : O_{2'}(N_x)| = |L : N_x|\,|N_x : O_{2'}(N_x)| < \infty .$$

$O_{2'}(N_x) \lhd L$ hence $O_{2'}(L) \supset O_{2'}(N_x)$ and so

$$|L : O_{2'}(L)| < \infty.$$

b) G is not simple.

Assume that G is simple with finite Sylow 2-subgroup S. For each involution x in G, the subgroup $C_G(x)$ is a proper subgroup and by the previous paragraph, $C_G(x)$ is almost locally solvable. The group G contains an elementary abelian 2-subgroup of order four. Otherwise there is a unique involution i in the centre of the Sylow 2-subgroup S of G. Since Sylow 2-subgroups are conjugate every Sylow 2-subgroup contains at most one conjugate of i, then by [3] Theorem (1.1.4) G is not simple. Hence we may assume that G contains an elementary abelian 2-subgroup V of order four. Let x_1, x_2, x_3 be the nontrivial involutions in V. Then

$$| C_G(x_i) : O_{2'}(C_G(x_i)) | < \infty \quad i = \mathrm{i}, 2, 3.$$

Since S is finite, the 2-rank of G is finite. Then again by [1] Theorem 9

$$| G : < O_{2'}(C_G(x_i)) : i = 1, 2, 3 > | < \infty.$$

Since our group does not have a subgroup of finite index

$$G = < C_G(x_i) : i = 1, 2, 3 >.$$

But again $C_G(x_i)$ is proper subgroup of G for all $i = 1, 2, 3$. But by [5] Lemma 2.10 G cannot be generated by two proper subgroups. Hence $G = C_G(x_i)$ for some $i = 1, 2, 3$ which is impossible since $x_i \notin Z(G) = 1$. So G is not simple.

Since we have non-trivial normal subgroups either G has a maximal normal subgroup or G is a union of an ascending series of proper normal subgroups N_i. In the latter case there exists i such that $S \subseteq N_i \lhd G$ and by a Frattini argument

$$G = N_i N_G(S).$$

But G cannot be generated by two proper subgroups, and N_i is a proper subgroup so $G = N_G(S)$. Hence S is a normal subgroup of G. The group S is finite and normal whence [5] Lemma 2.2 implies $S \leq Z(G)$. Since S is finite abelian and a maximal 2-subgroup, G/S is a $2'$-group. Let Σ be a local system consisting of finite subgroups and containing S. We can find such a local system since G is countable by [5] Lemma 2.14 and S is finite. Any element K_i in the local system is a finite subgroup of G containing S and $(|K_i/S|, |S|) = 1$. Then by the Schur-Zassenhaus theorem $K_i = S \times L_i$ as $S \leq Z(G)$. The group L_i is a $2'$-group. But this is true for all $K_i \in \Sigma$. Since the complements L_i of S are unique by embedding for each i, $L_i < L_{i+1}$ we get

$$G = S \times O_{2'}(G).$$

Since S is finite and G does not have a subgroup of finite index $G = G_{2'}(G)$ which is impossible since there exists nontrivial $x \in G$ such that $2 \mid o(x)$.

It remains to show the first possibility, that G contains a maximal normal subgroup is impossible. If there exists a maximal normal subgroup N, then G/N is a simple group satisfying min-2. By [5] Lemma 2.4 G/N is barely transitive and by the first paragraph a barely transitive locally finite group satisfying min-2 cannot be simple.

This proof also says that in a locally finite barely transitive group all maximal 2-subgroups are infinite and indeed not *Černikov.*

Proposition 2. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If for a fixed prime p every p-subgroup of H is solvable, then G is a union of proper normal subgroups. In particular G is not simple.*

Proof. Assume if possible that, G is a locally finite barely transitive simple group. Let P be a maximal p-subgroup of G. Bare transitivity of G implies that $|P : P \cap H| < \infty$. The subgroup $P \cap H$ is a p-subgroup of H and hence contained in a maximal p-subgroup of H. But maximal p-subgroups of H are solvable. Therefore $P \cap H$ is a solvable p-group. By bare transitivity we have $|P : P \cap H| < \infty$ which implies that P is solvable. Therefore every p-subgroup of G is solvable. Every locally finite simple group is either linear or non-linear. But a non-linear locally finite simple group contains finite p-subgroups of arbitrary derived length. Hence G cannot be a non-linear group. Then G is a linear group, but we show in [5] Lemma 2.11 that a locally finite barely transitive group cannot be a group of Lie type.

Let N be a proper normal subgroup of G. If N is a maximal normal subgroup of G, then G/N is a simple barely transitive group with HN/N its solvable point stabilizer. Hence there exists no maximal normal subgroup and G is a union of its proper normal subgroups.

Proposition 3. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If H is locally solvable, then G is a union of proper normal subgroups. In particular G is not simple.*

Proof. If G is locally solvable, then G cannot be a simple group as the only locally finite-solvable simple groups are finite cyclic groups.

Let K be a proper subgroup of G. Then $|K : K \cap H| < \infty$. So K has a locally solvable subgroup of finite index. Hence every proper subgroup of G is almost locally solvable. Then by [4] the only locally finite simple groups having each proper subgroup is almost locally solvable are either linear group A_1 or ${}^2B_2$. But these groups cannot be isomorphic to a barely transitive group [5] Lemma 2.11. One can show easily as in the Proposition 2 that there exists no maximal normal subgroup of G. Hence G can be written as union of its proper normal subgroups.

Proposition 4. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If a proper subgroup X of G involves an infinite simple group, such that* $Y \lhd X$ *and* X/Y *isomorphic to an infinite simple group, then* *a*) *Y cannot be locally solvable.*

b) *Y cannot be finite.*

c) *H involves an infinite simple group.*

Proof. a) Assume if possible that Y is locally solvable and X/Y is infinite simple. Since each proper subgroup of G is residually finite X is residually finite. Then for all $1 \neq x \in X$ we have $N_x \lhd X$ such that $x \notin N_x$ and $|X : N_x| < \infty$. But then $N_x Y/Y \trianglelefteq X/Y$. Since X/Y is infinite simple we have either $N_x Y = Y$ or $N_x Y = X$. Assume if possible that there exists $1 \neq x \in X$ such that $N_x Y = Y$. Then $N_x \leq Y$. But then $|X : Y| < |X : N_x| < \infty$ which is impossible. Hence we have $N_x Y = X$ for all $1 \neq x \in X$. Then $Y/(Y \cap N_x) \cong (Y N_x)/N_x = X/N_x$. Finiteness of $| X/N_x |$ and locally solvableness of Y implies that, there exist $n_x \in N$ satisfying $X^{(n_x)} \leq N_x$ for all $x \in X$. If there exists an upper bound m for the set $I = \{n_x \mid 1 \neq x \in X\}$, then $X^{(m)} \leq N_x$ for all $1 \neq x \in X$. Hence $X^{(m)} \leq \bigcap_{x \in X} N_x = 1$ i.e. X is solvable which is not the case. Hence we may assume that there exists no upper bound for the set I. But then $X^{(n_x)} \leq N_x$ hence $\bigcap_{n_x \in I} X^{(n_x)} \subset \bigcap_{x \in X} N_x = 1$. But this implies X is locally solvable which is impossible. Indeed let $A = \langle x_1, x_2, \ldots, x_t \rangle$ be a finite subgroup of X. Then consider $A^{(1)}, A^{(2)}, \ldots$. If A is not solvable, then there exists $k \in N$ such that $1 \neq A^{(k)} = A^{(k+1)} = \ldots$. But then $A^{(k)} \leq \bigcap_{n_x \in I} X^{(n_x)} = 1$. Hence A is solvable. This proves (a).

b) If Y is finite then by residual finiteness of X, there exists a normal subgroup N_Y of X such that $N_Y \cap Y = 1$ and X/N_Y has finite order. Then

$$N_Y Y/Y \trianglelefteq X/Y.$$

But X/Y is infinite simple. Hence $N_Y Y = X$, so $N_Y \cong N_Y/N_Y \cap Y \cong N_Y Y/Y = X/Y$. The group N_Y is residually finite hence finiteness of Y is impossible.

c) By bare transitivity for each proper subgroup X of G we have $|X : X \cap H| < \infty$, so there exists $K \leq X \cap H$ such that $K \lhd X$ and $|X : K| < \infty$. Then $K Y/Y \lhd X/Y$. Since X/K is finite and X/Y infinite simple, then $K Y = X$. But $K/K \cap Y \cong K Y/Y = X/Y$ and $K \cap Y \leq X \cap H \cap Y \leq H \cap Y$. Hence $K \leq H$ and involves the infinite simple group $K/(K \cap Y)$.

So in case of H is locally solvable, G does not have a proper subgroup X which involves an infinite simple group.

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