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SOME OLD AND RECENT ARITHMETICAL RESULTS CONCERNING MODULAR FORMS AND RELATED ZETA FUNCTIONS

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I. NEAR HOLOMORPHY AND ARITHMETICITY

Our first topic is the arithmeticity of the values of a Γ-invariant function, not necessarily meromorphic, on a hermitian symmetric space $\mathscr{H}$, where Γ is an arithmetic discontinuous group acting on $\mathscr{H}$. We start with the simplest case in which $\mathscr{H}$ is the upper half plane

$$H = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$$

and Γ is a *congruence subgroup* of $SL_2(\mathbf{Z})$. Here for any commutative ring A with identity element we denote by $SL_2(A)$ the group of all (2×2)-matrices of determinant 1 with entries in A, and call a subgroup of $SL_2(\mathbf{Z})$ a *congruence subgroup* if it contains

$$\left\{\alpha \in SL_2(\mathbf{Z}) \;\middle|\; \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\right\}$$

for some positive integer N. For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R})$ and $z \in H$ we put

$$\alpha(z) = (az + b)/(cz + d), \tag{1.1a}$$

$$j_\alpha(z) = cz + d. \tag{1.1b}$$

Further for $k \in \mathbf{Z}$ and a function $f: H \to \mathbf{C}$ we define $f\|_k \alpha : H \to \mathbf{C}$ by

$$(f\|_k \alpha)(z) = j_\alpha(z)^{-k} f(\alpha(z)) \qquad (z \in H). \tag{1.2}$$

Given a congruence subgroup Γ of $SL_2(\mathbf{Z})$, we call a holomorphic function f on H a (holomorphic) *modular form of weight k with respect to Γ* if the following two conditions are satisfied :

$$f\|_k \gamma = f \quad \textit{for every } \gamma \in \Gamma. \tag{1.3a}$$

For every $\alpha \in SL_2(\mathbf{Z})$ *one has* $(f\|_k \alpha)(z) = \sum_{n=0}^{\infty} c_{\alpha n} . \exp(2\pi i n z/N_\alpha)$ *with* $c_{\alpha n} \in \mathbf{C}$ and $0 < N_\alpha \in \mathbf{Z}$. (1.3b)

The last condition implies in particular that

$$f(z) = \sum_{n=0}^{\infty} c_n \,.\, \exp(2\pi i n z/N)$$

with $c_n \in \mathbf{C}$ and $0 < N \in \mathbf{Z}$. Given a subfield L of $\mathbf{C}$, we say that f is *L-rational* if $c_n \in L$ for ail n.

A *modular function with respect to* Γ is a meromorphic function φ on H such that $\varphi = f/g$ with two modular forms f and g of the same weight with respect to Γ. We call φ L-rational if we can choose L-rational f and g. We are now ready to state the main theorem of complex multiplication, due to Kronecker and Weber, in a simplified form :

Theorem 1.0. Let K be an imaginary quadratic field embedded in $\mathbf{C}$, and φ a $\mathbf{Q}_{ab}$-rational modular function with respect to a congruence subgroup of $SL_2(\mathbf{Z})$. Then $\varphi(\tau)$, if finite, belongs to K_{ab} for every $\tau \in H \cap K$.

Here for a subfield L of $\mathbf{C}$ we denote by L_{ab} the maximal abelian extension of L in $\mathbf{C}$. In fact, an explicit law of reciprocity for the extension $K(\varphi(\tau))/K$ can be given in the sense that if $\sigma = \left(\dfrac{K(\varphi(\tau))/K}{\mathfrak{P}}\right)$ with a prime $\mathfrak{P}$ of K unramified in $K(\varphi(\tau))$, then $\varphi(\tau)^\sigma = \varphi'(\tau)$ with φ' determined by φ, τ, and $\mathfrak{P}$. In a certain case, $\varphi(\tau)^\sigma = \varphi(\tau')$ with τ' determined by τ and $\mathfrak{P}$. This is so if φ is the standard modular function $J = g_2^3/(g_2^3 - 27g_3^2)$, in which case $K(J(\tau))$ is the Hilbert class field over K provided $\mathbf{Z} + \mathbf{Z}\tau$ is a fractional ideal in K.

Our aim is to answer the following two questions :

(I) *Can one obtain a similar result with non-meromorphic functions in place of* φ?

(II) *Can one generalize the above theorem (as well as its non-meromorphic version) to the higher-dimensional ease?*

Let us first answer (I). We do this by introducing the notion of near holomorphy. A function $f\colon H \to \mathbf{C}$ is called *nearly holomorphic* if it is of the form $f(z) = \sum_{m=0}^{M} y^{-m} p_m(z)$ with holomorphic functions p_m on H, $0 \leq M \in \mathbf{Z}$, and $y = \mathrm{Im}(z)$. It can easily be shown that f has such an expression if and only if $(y^2\, \partial/\partial \bar{z})^{M+1} f = 0$. Now we call such an f *nearly holomorphic modular form of weight k with respect to* Γ if it satisfies (1.3a) and the following condition :

For every $\alpha \in SL_2(\mathbf{Z})$ *one has*

$$(f\|_k \alpha)(z) = \sum_{m=0}^{M} (\pi y)^{-m} \sum_{n=0}^{\infty} c_{\alpha mn} \exp(2\pi i n z/N_\alpha)$$

with $c_{\alpha mn} \in \mathbf{C}$ *and* $0 < N_\alpha \in \mathbf{Z}$. (1.4)

We call f *L-rational* if $c_{1mn} \in L$ for all m and n. Then we have :

Theorem 1.1. The assertion of Theorem 1.0 is true for $\varphi = f/g$ with $\mathbf{Q}_{ab}$-rational nearly holomorphic modular forms f and g of the same weight.

Nearly holomorphic forms can be obtained in the following way: Define a differential operator δ_k^p for $0 \leq p \in \mathbf{Z}$ by

$$\delta_k^p = \delta_{k+2p-2} \cdots \delta_{k+2}\, \delta_k, \quad \delta_k = \frac{1}{2\pi i}\left(\frac{k}{2iy} + \frac{\partial}{\partial z}\right). \tag{1.5}$$

Then we can show that *if* f *is a holomorphic modular form of weight* k, *then* $\delta_k^p f$ *is a nearly holomorphic modular form of weight* $k + 2p$. *If* f *is L-rational, so is* $\delta_k^p f$. There is also a naturally defined nearly holomorphic form obtained from an *Eisenstein series*

$$E_m(z, s) = (1/2) \sum (cz + d)^{-m} |cz + d|^{-2s}, \tag{1.6}$$

where $0 < m \in 2\mathbf{Z}$, $s \in \mathbf{C}$, and (c, d) runs over all relatively prime pairs of integers. This is convergent for $\mathrm{Re}(2s + m) > 2$; moreover if we put

$$E_m^*(z, s) = \zeta(2s + m)\, E_m(z, s), \tag{1.7}$$

then we obtain :

Proposition 1.2. There is a function on $H \times \mathbf{C}$, real analytic in z and holomorphic in s, which coincides with $\Gamma(s+m) E_m^*(z,s)$ for $\mathrm{Re}(2s+m) > 2$. Moreover, given a compact subset A of $\mathbf{C}$, one has $|\Gamma(s+m) E_m^*(z,s)| \leq by^c$ if $s \in A$ and $y > 1$, where b and c are positive constants depending only on A.

Thus we can speak of $E_m(z, 0)$ for every such m, including the case $m = 2$. Obviously $E_m(z, 0)$ is holomorphic in z if $m > 2$; in fact, it is a modular form of weight m. If $m = 2$, however, we have

$$E_2(z, 0) = \frac{-3}{\pi y} + 1 - 24 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d\right) e^{2\pi i n z},$$

and this is a nearly holomorphic modular form of weight 2. We can prove :

Proposition 1.3. Every nearly holomorphic modular form of weight k is of the form

$$f(z)=\sum_{0\le p\le k/2}\delta_{k-2p}^{p}\,g_p+\begin{cases}c\delta_2^{(k/2)-1}E_2(z,0) & \text{if } k\in 2\mathbf{Z},\\ 0 & \text{if } k\notin 2\mathbf{Z},\end{cases}\tag{1.8}$$

where g_p is a holomorphic modular form of weight $k-2p$ and $c\in\mathbf{C}$.

We note here an interesting fact :

Proposition 1.4. The function $\pi^{-m-t}E_m^*(z,t)$ is a $\mathbf{Q}$-rational nearly holomorphic modular form of weight m for every $t\in\mathbf{Z}$ such that $-m<t\le 0$.

Let us now turn to the higher-dimensional case. We restrict our exposition to the case in which the group is $Sp(n,\mathbf{Z})$, where

$$Sp(n,A)=\{\alpha\in M_{2n}(A)\mid {}^t\alpha J\alpha=J\},\quad J=\begin{pmatrix}0 & -1_n\\ 1_n & 0\end{pmatrix},$$

for any commutative ring A with identity element, and the space is

$$H_n=\{z\in M_n(\mathbf{C})\mid t_z=z \text{ and } \mathrm{Im}(z) \text{ is positive definite}\}$$

(The symbol $M_m(A)$ denotes the set of all $(m\times m)$-matrices with entries in A). For $\alpha=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in Sp(n,\mathbf{R})$ with a,b,c,d of size n and $z\in H_n$ we put

$$\alpha(z)=(az+b)(cz+d)^{-1},\tag{1.9a}$$

$$j_\alpha(z)=\det(cz+d).\tag{1.9b}$$

For $k\in\mathbf{Z}$ and $f\colon H_n\to\mathbf{C}$ we can define $f\|_k\alpha\colon H_n\to\mathbf{C}$ by (1.2) with H_n in place of H. Given a congruence subgroup Γ of $Sp(n,\mathbf{Z})$ (which can be defined in the same fashion as in the case $n=1$), we call a holomorphic function f on H_n a *Siegel modular form of weight k with respect to Γ* if it satisfies (1.3a) and the following condition :

For every $\alpha\in Sp(n,\mathbf{Z})$ one has (1.10)

$$(f\|_k\alpha)(z)=\sum_{h\in S}c_{\alpha,h}\cdot\exp(2\pi i\cdot\mathrm{tr}(hz)/N_\alpha)$$

with $c_{\alpha h}\in\mathbf{C}$ and $0<N_\alpha\in\mathbf{Z}$, where

$$S=\{h\in M_n(\mathbf{Z})\mid h \textit{ is symmetric and positive definite}\}$$

(In fact, (1.10) is automatically satisfied if $n>1$). Then a *Siegel modular function* is a meromorphic function $\varphi=f/g$ with Siegel modular forms f and g of the same weight. The L-rationality can be defined in the same fashion as in the case $n=1$.

To generalize Theorem 1.0, we take a *CM-field of degree* $2n$, by which we mean a totaly imaginary quadratic extension K of a totally real algebraic number field F of degree n. Then we can always find a ring-injection $\lambda : K \to M_{2n}(\mathbf{Q})$ such that $\lambda(a^\rho) = J \, . \, {}^t\lambda(a) J^{-1}$, where p is the generator of $\mathrm{Gal}(K/F)$. Given such a λ, we see that $\lambda(\{\alpha \in K \mid \alpha\alpha^\rho = 1\})$ is a subgroup of $Sp(n, \mathbf{Q})$ and has a unique common fixed point τ on H_n. Moreover, there is a ring-injection $\mu : K \to M_n(\mathbf{C})$ determined by

$$\lambda(a) \begin{pmatrix} \tau \\ 1_n \end{pmatrix} = \begin{pmatrix} \tau \\ 1_n \end{pmatrix} \mu(a) \qquad (a \in K). \tag{1.11}$$

Let K' be the field generated over $\mathbf{Q}$ by $\mathrm{tr}(\mu(a))$ for all $a \in K$. Then K' is a CM-field whose degree may or may not be equal to $[K : \mathbf{Q}]$. In this setting a generalization of Theorem 1.0 can be given as follows :

Theorem **1.5**. Let φ be a $\mathbf{Q}_{ab}$-rational Siegel modular function. Then $\varphi(\tau)$, if finite, belongs to K'_{ab}.

The reciprocity-law can also be given in this case.

We call $f : H_n \to \mathbf{C}$ *nearly holomorphic* if it is a polynomial in the entries of y^{-1} with holomorphic functions as coefficients, where $y = \mathrm{Im}(z)$. Such an f is called a *nearly holomorphic modular form of weight* k *with respect to* Γ if it satisfies (1.3a) and the condition

For every $\alpha \in Sp(n, \mathbf{Z})$ *one has*

$$(f \|_k \alpha)(z) = \sum_{m=1}^{M} p_{\alpha m}(\pi^{-1} y^{-1}) \sum_{h \in S} c_{\alpha m h} \, . \, \exp(2\pi i \, . \, \mathrm{tr}(hz)/N_\alpha) \tag{1.12}$$

with $c_{\alpha m h} \in \mathbf{C}$, $0 < N_\alpha \in \mathbf{Z}$, *and polynomials* $p_{\alpha m}(\pi^{-1} y^{-1})$ *in the entries of the matrix* $\pi^{-1} y^{-1}$.

We call f *L-rational* if c_{1mh} and the coefficients of p_{1m} belong to L for all m and h. Then we have:

Theorem **1.6**. The assertion of Theorem 1.4 is true for $\varphi = f/g$ with $\mathbf{Q}_{ab}$-rational nearly holomorphic modular forms f and g of the same weight.

The generalizations of (1.5), (1.6), and (1.7) are given by

$$\Delta_k^p = (\pi i)^{-np} \det(y)^{(n+1)/2-k-p}$$
$$. \det\left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial z_{ij}}\right)^p \det(y)^{k+p-(n+1)/2}, \tag{1.13}$$

$$E_m(z, s) = \det(y)^s \sum_{\alpha \in T} j_\alpha(z)^{-k} \, | j_\alpha(z) |^{-2s}, \tag{1.14a}$$

$$E_m^*(z,s)=E_m(z,s)\,\zeta(2s+m)\prod_{i=1}^{[n/2]}\zeta(4s+2m-2i), \tag{1.14b}$$

where $T=\Gamma_P-\Gamma$, $\Gamma_P=\left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\in\Gamma\right\}$. It can be shown that (i) if f is an L-rational holomorphic or nearly holomorphic modular form of weight k, then $\Delta_k^p f$ is an L-rational nearly holomorphic modular form of weight $k+2p$; (ii) if n is odd, $E_m^*(z,t)$ is nearly holomorphic for every integer t such that $(n+1)/2-m\le t\le 0$ with a few exceptions; a similar but somewhat different type of result holds for even n.

Nearly holomorphic functions can be defined on an arbitrary complex Kähler manifold. Let Ω be a fundamental 2-form on such a manifold V of complex dimension n. Every point of V has a coordinate neighborhood U in which $\Omega=i\partial\bar\partial\varphi$ with a real-valued function φ on U. Put $\Omega=i\sum_{p,q=1}^{n}h_{pq}\,dz_q\wedge d\bar z_p$ with complex coordinates $z_1,\dots,z_n$ on U; define n vector fields $X_1,\dots,X_n$ by $\sum_{q=1}^{n}h_{pq}X_q=\partial/\partial\bar z_p$. Then *a function on U is annihilated by all monomials of the X_i of degree r if and only if it is a polynomial in $\partial\varphi/\partial z_1,\dots,\partial\varphi/\partial z_n$ of degree $<r$ with holomorphic functions on U as coefficients.* We call a C^∞ function on V *nearly holomorphic* if its restriction to every such U is such a polynomial. If $V=H_n$, we can take $\varphi=-\log(\det(\mathrm{Im}(z)))$, and find that this definition is consistent with the previous one.

Among general Kähler manifolds, hermitian symmetric spaces form the class of manifolds on which nearly holomorphic functions appear most naturally. Any hermitian symmetric space V of noncompact type can be given as $V=G/K$ with a semisimple noncompact group G and its maximal compact subgroup K. Let $\mathfrak{G}$ and $\mathfrak{T}$ be their Lie algebras, and $\mathfrak{G}_{\mathbf{C}}=\mathfrak{T}_{\mathbf{C}}\oplus\mathfrak{P}_+\oplus\mathfrak{P}_-$ the well known decomposition such that $\mathfrak{P}_+$ and $\mathfrak{P}_-$ correspond to holomorphic and antiholomorphic tangent vectors on V at the origin. Then a nearly holomorphic function on V corresponds exactly to a function on G annihilated by all homogeneous elements of a fixed degree in the symmetric algebra over $\mathfrak{P}_-$. If Γ is an arithmetic discrete subgroup of G, then we can speak of nearly holomorphic automorphic forms on V with respect to Γ, and the results similar to Theorems 1.5 and 1.6 can be obtained.

II. SPECIAL VALUES OF SOME ZETA FUNCTIONS

Let $\Gamma=SL_2(\mathbf{Z})$. Though all the results in this section can be extended to the case of modular forms with respect to congruence subgroups of Γ, we consider for simplicity only those of level 1, that is, those with respect to Γ. Let M_k denote the set of all modular forms of level 1 and weight k. We assume that $0 < k \in 2\mathbf{Z}$, since $M_k = \{0\}$ for odd k and $M_0 = \mathbf{C}$. We call an element $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ of M_k a *cusp form* if $a_0 = 0$, and denote by S_k the subset of M_k consisting of the cusp forms. As an example we mention

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \qquad (q = e^{2\pi i z}),$$

which is an element of S_{12} (In fact, $S_k = \{0\}$ for $k < 12$). For two elements f and g of M_k we define their inner product $\langle f, g \rangle$ by

$$\langle f, g \rangle = \int_{\Phi} f(z) \overline{g(z)} y^k d_H z \tag{2.1}$$

whenever the integral is convergent, which is the case if f or g is a cusp form. Here $z = x + iy$ as usual, $\Phi = \Gamma - H$, and $d_H z$ is the invariant measure $y^{-2} dx\, dy$ on H. It is known that

$$M_k = S_k \oplus \mathbf{C} E_k^0 \qquad (k > 2), \tag{2.2a}$$

$$\langle S_k, E_k^0 \rangle = 0, \tag{2.2b}$$

where $E_k^0 = E_k(z, 0)$ with the series $E_k(z, s)$ of (1.6).

We associate a Dirichlet series $D_f(s)$ to each $f \in M_k$ by

$$D_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}. \tag{2.3}$$

This is convergent for sufficiently large $\mathrm{Re}(s)$, and can be continued to the whole s-plane as a meromorphic function. In fact, if we put

$$R_f(s) = (2\pi)^{-s} \Gamma(s) D_f(s), \tag{2.4}$$

then R_f is holomorphic everywhere except for possible simple poles at $s = 0$ and $s = k$, and satisfies

$$R_f(k - s) = i^k R_f(s). \tag{2.5}$$

The residue of R_f at $s = 0$ is $-a_0$, and hence R_f is entire if $f \in S_k$.

We are interested in the arithmetic nature of the values $D_f(m)$ for some integers m when f is a *cusp form*, an assumption we make throughout the rest of the paper. Since R_f is entire and $\Gamma(s)$ has a pole at every $m \in \mathbf{Z}, \leq 0$, we see from (2.4) that $D_f(m) = 0$ for such an m. Therefore, in view of (2.5), it is natural to consider $D_f(m)$ for an integer m belonging to the open interval $(0, k)$, which may be called the *critical interval* for D_f.

To state the result on $D_f(m)$, we need the notion of Hecke eigenform. Without details, let us merely say that one can define a $\mathbf{C}$-linear endomorphism T_n on S_k, called a *Hecke operator*, for each $n \in \mathbf{Z}, \geq 0$. Moreover the T_n are hermitian with respect to (2.1) and form a commutative ring. Thus S_k is spanned by their common eigenfunctions, which we call *Hecke eigenforms*. If $f(z) = \sum_{n=1}^{\infty} a_n n^{-s}$ is a Hecke eigenform, then $a_1 \neq 0$. We call it *normalized* if $a_1 = 1$. For a normalized Hecke eigenform we know that $T_n(f) = a_n f$ for every n and D_f has an Euler product :

$$D_f(s) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}. \tag{2.6}$$

For $h(z) = \sum_{n=0}^{\infty} c_n e^{2\pi i n z} \in M_k$ let K_h denote the field generated over $\mathbf{Q}$ by the c_n. If f is a normalized Hecke eigenform, then K_f is a totally real algebraic number field of finite degree. Clearly $K_f = \mathbf{Q}$ if $f = \Delta$. Now we have:

Theorem 2.1. If f is a normalized Hecke eigenform, then $R_f(m)/R_f(n) \in K_f$ for every two integers m and n inside the critical interval such that $m \equiv n \pmod 2$.

Postponing the proof, let us introduce another type of Dirichlet series

$$D(s; f\ g) = \sum_{n=1}^{\infty} a_n b_n n^{-s}. \tag{2.7}$$

Here f and a_n are the same as above and $g(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z} \in M_l$ with $l < k$. At first sight this may look artificial, but in fact it has been observed in recent years by several researchers that the series of type (2.7) are natural arithmetical objects which can be extended to the higher-dimensional case without losing their good arithmetic properties, while the higher-dimensional analogues of (2.3) have no such advantages.

To express (2.7) by an integral, let us assume for simplicity that a_n and b_n are all real, since the general case can easily be reduced to this special case. Observing that

$$f(z)\overline{g(z)} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_m b_n e^{2\pi i(m-n)x} e^{-2\pi(m+n)y},$$

we obtain

$$\int_0^1 f(z)\overline{g(z)}\, dx = \sum_{n=1}^{\infty} a_n b_n e^{-4\pi n y},$$

and hence

$$\int_0^\infty \int_0^1 f(z)\overline{g(z)}\,dx\,y^{s-1}\,dy = (4\pi)^{-s}\,\Gamma(s)\,D(s;f,g). \tag{2.8}$$

Let $\Gamma_\infty = \left\{\pm\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix} \middle| m \in \mathbf{Z}\right\}$ and $\Psi = \Gamma_\infty - H$. Since $\{x+iy \mid 0 \le x < 1\}$ represents Ψ, the left-hand side of (2.8) can be written $\int_\Psi f\bar{g}y^{s+1}\,d_H z$. Let A be a complete set of representatives for $\Gamma_\infty - \Gamma$. Then $\bigcup_{\alpha\in A} \alpha\Phi$ represents $\Gamma_\infty - H$, and therefore

$$\int_\Psi f(z)\overline{g(z)}\,y^{s+1}\,d_H z = \sum_{\alpha\in A}\int_{\alpha\Phi} f(z)\overline{g(z)}\,y^{s+1}\,d_H z$$
$$= \int_\Phi \left\{\sum_{\alpha\in A} (f\bar{g}y^{s+1})\circ\alpha\right\} d_H z.$$

Since $(f\bar{g}y^{s+1})\circ\alpha = f\bar{g}y^{s+1}\,\overline{j_\alpha(z)}^{\,l-k}\,|j_\alpha(z)|^{-2-2s}$, the last integral over Φ can be written

$$\int_\Phi f\bar{g}y^{s+1-k}\sum_{\alpha\in A}\overline{j_\alpha(z)}^{\,l-k}\,|j_\alpha(z)|^{-2-2s}\,y^k\,d_H z = \langle f(z), g(z)\,E_{k-l}(z, \bar{s}+1-k)\rangle$$

with E_m of (1.6). By virtue of Proposition 1.2, we can show that the last inner product times $\zeta(2s+2-k-l)\,\Gamma(s+1-l)$ is convergent for every s. In this way we can show that if $0 < l < k$, then

$$(2\pi)^{-2s}\,\Gamma(s)\,\Gamma(s+1-l)\,\zeta(2s+2-k-l)\,D(s;f,g)$$

is entire and invariant under $s \to k+l-1-s$.

Theorem 2.2. Put $C(s) = \zeta(2s+2-k-l)\,D(s;f,g)$ with a *normalized Hecke eigenform* f. Then $C(t) \in \pi^{2t+1-l}\langle f,f\rangle K_f K_g$ for every integer t such that $l-1 < t < k$.

Notice that $C(m) = 0$ if $l-1 \ge m \in \mathbf{Z}$, and hence $(l-1, k)$ is the "critical interval" for C (In all my papers in *References* the inner product $\langle f, g\rangle$ was defined with an extra factor $1/(\int_\Phi d_H z)$ on the right-hand side of (2.1), and hence the above theorem differs from [8, Theorem 4] by that factor).

Let us now sketch the proof. The above computation shows that

$$(4\pi)^{-s}\,\Gamma(s)\,C(s) = \langle f(z), g(z)\,E^*_{k-l}(z, \bar{s}+1-k)\rangle$$

with E^*_m of (1.7). We evaluate this at $s = k-1$. Assuming that $k-l \ne 2$, we know that $\pi^{l-k}E^*_{k-l}(z,0)$ is a $\mathbf{Q}$-rational element of M_{k-l}. Therefore if we put $h(z) = \pi^{l-k}g(z)\,E^*_{k-l}(z,0)$, then h is a K_g-rational element of M_k, so that

$h = p + \lambda E_k^0$ with $p \in S_k$ and $\lambda \in \mathbf{C}$ by virtue of (2.2a). Writing p as a linear combination of Hecke eigenforms, we can put $p = \mu f + r$ with $\mu \in \mathbf{C}$ and an element r of S_k such that $\langle f, r \rangle = 0$. From the K_g-rationality of h, we can derive that p is K_g-rational and also that $\mu \in K_f K_g$. We have then

$$\begin{aligned}\pi^{l-k}(4\pi)^{1-k}\Gamma(k-1)\,C(k-1) &= \langle f, h\rangle \\ &= \langle f, \mu f + r + \lambda E_k^0\rangle = \mu\,\langle f, f\rangle\end{aligned}$$

by virtue of (2.2b), since $K_f K_g \subset \mathbf{R}$. This proves our theorem for $t = k - 1$ under the assumption that $k - l \neq 2$.

To treat the case $t < k - 1$, put $m = t + 1 - k$. By Proposition 1.4 $\pi^{l-k-m} E_{k-l}^*(z, m)$ is $\mathbf{Q}$-rational and nearly holomorphic, and hence by Proposition 1.3 we have $\pi^{l-k-m} g(z) E_{k-l}^*(z, m) = h + \delta_{k-2}\, r$ with $h \in M_k$ and a nearly holomorphic form r of weight $k - 2$. Now we can show that $\langle S_k, \delta_{k-2}\, q\rangle = 0$ for any slowly increasing C^∞ function q satisfying (1.3a) with $k - 2$ in place of k, in particular for any nearly holomorphic form q of weight $k - 2$. Therefore, writing the present h in the form $h = p + \lambda E_k^0$ as in the case $t = k - 1$, we obtain the desired result (including even the case $k - l = 2$), though the rationality question becomes somewhat more nontrivial.

Coming back to D_f, let us now derive Theorem 2.1 from Theorem 2.2. We first observe that

$$aE_l(z, 0) = by^{-1} + c + \sum_{n=1}^{\infty}\Big(\sum_{0<d|n} d^{l-1}\Big) e^{2\pi i n z} \tag{2.9}$$

with suitable constants a, b and c; $b \neq 0$ only when $l = 2$. Taking the function of (2.9) to be g of $D(s; f, g)$, we can derive from (2.7) that in this case $C(s) = D_f(s - l + 1)\, D_f(s)$. By Theorem 2.2 we have

$$\pi^{l-1-2t} D_f(t - l + 1)\, D_f(t) \in \langle f, f\rangle\, K_f \tag{2.10}$$

for every two integers l and t such that $0 < l \leq t < k$, $l \in 2\mathbf{Z}$ (Strictly speaking, the case $l = 2$ must be excluded, but it can actually be shown that (2.10) is true even in this case). Now the argument similar to the proof of nonvanishing of ζ on the line $\mathrm{Re}(s) = 1$ shows that $D_f(s) \neq 0$ for $\mathrm{Re}(s) \geq (k + 1)/2$. Since $k \geq 12$, we see that $D_f(k - 1)\, D_f(k - 2) \neq 0$. Now (2.10) shows that the following products belong to $\langle f, f\rangle\, K_f$:

$$\pi^{3-2k} D_f(k-2)\, D_f(k-1),$$

$$\pi^{1-m-k} D_f(m)\, D_f(k-1) \quad \text{for} \quad 0 < m < k-1, \ m \in 2\mathbf{Z},$$

$$\pi^{2-m-k} D_f(m)\, D_f(k-2) \quad \text{for} \quad 0 < m < k-2, \ m \notin 2\mathbf{Z}.$$

Dividing the second and third products by the first one, we obtain Theorem 2.1.

Theorems 2.1 and 2.2 can be generalized in several ways. First of all, with any Dirichlet character φ, one can consider

$$D(s; f, \varphi) = \sum_{n=1}^{\infty} \varphi(n)\, a_n\, n^{-s},$$

which includes $D_f(s)$ as a special case. Let K_φ denote the field generated over $\mathbf{Q}$ by the values of φ. Then we have :

Theorem 2.3. There exist two constants $p_+(f)$ and $p_-(f)$ depending only on f such that $(\pi i)^{-m} D(m; f, \varphi)$, for every positive integer $m < k$, belongs to $p_+(f)\, K_f K_\varphi$ or $p_-(f)\, K_f K_\varphi$ according as $\varphi(-1) = (-1)^m$ or $(-1)^{m-1}$.

This as well as Theorem 2.2 can be further generalized to the case of Hilbert modular forms. We can also consider $D(m; f, g)$ when f or g is a Hilbert modular form of half-integral weight. Such values, as well as the higher-dimensional versions of $D(m; f, \varphi)$ are closely related to the periods of differential forms.

BIBLIOGRAPHICAL NOTE

The classical theory of modular forms (in particular, the theory of Hecke operators and Eisenstein series) can be found in [1, 2, 3]. Theorem 1.0 is classical. Its modern formulation is given in [3, Chapter 6]. Theorem 1.5 is a special case of a more general result concerning canonical models of [5]. Near holomorphy was introduced in [9, 10]. Theorem 1.1 is essentially a reformulation of a special case of [6, Main Theorem II]. Theorem 1.6 follows from [10, Proposition 3.9], or rather from its proof modified in a suitable way. Proposition 1.3 is included in [10, Theorem 5.2], and Proposition 1.4, as well as its higher-dimensional version, in [10, Theorem 4.2] and [12, Proposition 5.2]. Theorem 2.1 in the special case $f = \Delta$ was first given in [4]. Its general case, Theorem 2.2, and Theorem 2.3 were obtained in [7, 8]. The most general case involving Hilbert modular forms of integral or half-integral weight is discussed in [11, 12], in which the papers on this topic published after 1978 are listed. The connection of special values with the periods of differential forms is investigated in [11].

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