TRANSCENDENCE PROBLEMS CONNECTED WITH DRINFELD MODULES

M. WALDSCHMIDT (*)

During the last few years several remarkable transcendence results concerning Drinfeld modules have been proved, mainly by Jing Yu (see the list of references at the end of this report). We start here by introducing a generalization, due to Y. Hellegouarch, of the notion of Drinfeld module. Next we give a short survey of some of the recent transcendence results in this context. Finally we give a short description of Anderson's t-motives.

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1. GENERALIZED DRINFELD MODULES

Let k be a commutative field, $Z = k[t]$ the ring of polynomials in one variable over k, $Q = k(t)$ its quotient field, which is the field of rational functions in one variable, and $R = k((1/t))$ the field of formal Laurent power series in $1/t$, which is the completion of k for the valuation defined by $-$ deg : each $z \in R$, $z \neq 0$ can be written in a unique way

$$z = a_d t^d + a_{d-1} t^{d-1} + \ldots + a_1 t + a_0 + \frac{a_{-1}}{t} + \ldots$$

with $a_i \in k$, $(i \leq d)$ and $a_d \neq 0$; we define deg $z = d$; z is said to be *monic* if $a_d = 1$.

We are interested in transcendence problems : an element of R is said to be *rational* if it belongs to Q, *algebraic* if it belongs to the algebraic closure $\bar{Q}$ of Q into R, and *transcendental* otherwise.

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Of course, the letters Z, Q, R are chosen in analogy with the classical notations $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ for the ring of rational integers, the field of rational numbers and the field of real numbers. By the way, the torsion subgroup $\mathbf{Q}^*_{\text{tors}}$ of the multiplicative group of $\mathbf{Q}$ has just two elements, the same number as for the multiplicative group $\mathbf{Z}^*$ of units of $\mathbf{Z}$; here the multiplicative group Z^* of Z is just k^*; if k is finite with q elements then this group has $q-1$ elements. This is why congruences modulo 2 in the complex situation are often replaced by congruences modulo $q-1$ when one works over (the ring of polynomials of) a finite field $\mathbf{F}_q$.

The special case $k = \mathbf{F}_q$ has been considered by Carlitz already in 1935; the main tool here is the Frobenius $z \to z^q$. One can also deal with the field of rational functions of a projective irreducible curve over $\mathbf{F}_q$ (in place of $\mathbf{F}_q(t)$, which corresponds to the projective line $\mathbf{P}_1$), but the generalization we are considering now (due to Y. Hellegouarch) is of a different kind and works also in zero characteristic: we start with any continuous field endomorphism σ of R which induces the identity on k. Such an endomorphism is produced by substituting to t an element $\sigma(t)$ of R of degree $d \geq 1$:

$$\sigma(t) = c_d t^d + c_{d-1} t^{d-1} + \ldots \quad (c_j \in k).$$

We define $d = \deg \sigma$. Carlitz situation corresponds to $k = \mathbf{F}_q$, $\sigma(t) = t^q$ and $d = q$.

For $n \geq 0$ and $b_0, b_1, \ldots, b_n$ in R, we get a k-linear endomorphism $P = b_0 \sigma^0 + b_1 \sigma^1 + \ldots + b_n \sigma^n$ (with $\sigma^0 = 1_R$, $\sigma^1 = \sigma$, $\sigma^n = \sigma^{n-1} \circ \sigma$) which sends $z \in R$ to

$$b_0 z + b_1 \sigma(z) + \ldots + b_n \sigma^n(z) \in R.$$

We write also z^σ for $\sigma(z)$, hence

$$z^{\sigma^i} = \sigma^i(z) \quad \text{and} \quad P(z) = b_0 z + b_1 z^\sigma + \ldots + b_n z^{\sigma^n}.$$

Let $R\{\sigma\}$ be the ring of these endomorphisms. For $b \in R$ we denote also by b the endomorphism of multiplication by b (namely $b\sigma^0$). Since $\sigma(bz) = b^\sigma \sigma(z)$ we can write $\sigma b = b^\sigma \sigma$. Hence $R\{\sigma\}$ is a twisted ring of polynomials in σ, with the product given by the rule

$$\left(\sum_{i=0}^{n} b_i \sigma^i\right)\left(\sum_{j=0}^{m} c_j \sigma^j\right) = \sum_{h=0}^{m+n}\left(\sum_{i+j=h} b_i c_j^{\sigma^i}\right)\sigma^h.$$

Our next aim is to define a Z-module structure on R, i.e. a homomorphism $Z \times R \to R$, which induces a map φ from Z into the ring of endomorphisms of R. We require

— that the image of φ be in $R\{\sigma\}$,

— that φ be a k-homomorphism,

— and that, for $M \in Z$, the constant term of $\varphi(M)$ be M:

$$\varphi(M) = M\sigma^0 + M_1\sigma^1 + \dots + M_n\sigma^n .$$

Definition. A *generalized Drinfeld module* is a k-homomorphism $\varphi: Z \to R\{\sigma\}$ such that $\varphi(\alpha) = \alpha\sigma^0$ for $\alpha \in k$ and

$$\varphi(M) \equiv M\sigma^0 \mod (R\{\sigma\}\sigma) \quad \text{for all} \quad M \in Z.$$

Let us write the image of t:

$$\varphi(t) = t\sigma^0 + a_1\sigma + \dots + a_n\sigma^n \in R\{\sigma\}$$

with $a_n \neq 0$; then n is the *rank of* φ. The associated Z-module structure on R is given by

$$Z \times R \longrightarrow R$$
$$(M, x) \longrightarrow \varphi(M)x.$$

When K is a subfield of R containing Q, we say that φ is *defined over* K if, for all $M \in Z$, the coefficients M_i of $\varphi(M)$ belong to K.

Example. If we set $\varphi(t) = t\sigma^0 - \sigma$, we get the (generalized) *Carlitz module*. We shall denote it by γ.

We need now to introduce special functions. Let us denote by $R\{\{\sigma\}\}$ the σ-adic completion of $R\{\sigma\}$. Here is the exponential map:

Theorem 1 [HI]. Let φ be a generalized Drinfeld module of rank n:

$$\varphi(t) = t\sigma^0 + a_1\sigma + \dots + a_n\sigma^n \in R\{\sigma\}.$$

There exists a unique element

$$e_\varphi = \sigma^0 + b_1\sigma + \dots + b_m\sigma^m + \dots \in R\{\{\sigma\}\}$$

such that

$$e_\varphi\, t\,\sigma^0 = \varphi(t)\, e_\varphi .$$

For all $M \in Z$ one has $e_\varphi M\sigma^0 = \varphi(M)e_\varphi$. Further, if we set

$$[h] = \sigma^h(t) - t, \quad \text{and} \quad F_h = [h]\,\sigma([h-1]) \dots \sigma^{h-1}([1]), \quad (h \geq 1),$$

then $F_h b_h$ belongs to $Z\{\sigma\}[a_1, \dots, a_n]$. Furthermore, if we assume $\deg \sigma \geq 2$, then e_φ induces a k-linear continuous map $e_\varphi: R \to R$ such that

$$e_\varphi(z) = z + \sum_{h \geq 1} b_h z^{\sigma^h}.$$

Example. In the case $\varphi = \gamma$ (generalized Carlitz module), one gets $F_h b_h = (-1)^h$ and

$$e_\gamma = \sigma^0 - \frac{\sigma}{F_1} + \frac{\sigma^2}{F_2} - \dots .$$

Next we introduce the logarithm. We first set

$$a = \max_{1\leq i\leq n} \{\deg a_i\}, \quad d = \deg \sigma$$

and then we define $B \subset R$ by

$$B = \begin{cases} \left\{ z \in R \,;\ \deg z < 1 - \dfrac{a}{d-1} \right\} & \text{if } \varphi \text{ is of rank } n > 1, \\ \left\{ z \in R \,;\ \deg z < \dfrac{d-a}{d-1} \right\} & \text{if } \varphi \text{ is of rank } n = 1. \end{cases}$$

Theorem 2 [H1]. Under the assumption of Theorem 1, there exists a unique $\log_\varphi \in R\{\{\sigma\}\}$ such that

$$t \log_\varphi = \log_\varphi \varphi(t).$$

This element $\log_\varphi$ satisfies

$$e_\varphi \log_\varphi = \log_\varphi e_\varphi = \sigma^0 .$$

Further, if we set

$$L_h = [h]\,[h-1] \dots [1], \quad (h \geq 1)$$

and

$$\log_\varphi = \sigma^0 + c_1 \sigma + \dots + c_m \sigma^m + \dots$$

then $L_h c_h$ belongs to $Z\{\sigma\}[a_1, \dots, a_n]$. Furthermore, if $d = \deg \sigma \geq 2$, then $\log_\varphi$ induces a k-linear continuous map $\log_\varphi : B \to R$ such that

$$\log_\varphi(z) = z + \sum_{h\geq 1} c_h z^{\sigma^h}.$$

Let us restrict now our attention to the special case of the generalized Carlitz module γ. The exponential is

$$e_\gamma = \sigma^0 - \frac{\sigma}{F_1} + \frac{\sigma^2}{F_2} - \dots$$

while the logarithm is

$$\log_\gamma = \sigma^0 + \frac{\sigma}{L_1} + \frac{\sigma^2}{L_2} + \dots .$$

In this case $a = 0$, $n = 1$, $d = q$ and

$$B = \left\{ z \in C \,;\ \deg z < \frac{q}{q-1} \right\} .$$

We also assume that the characteristic of k does not divide $d - 1$ and that $\sigma(t)$ is monic.

Theorem 3 [H1]. There exists $\pi \in R(t^{1/(d-1)})$ (defined modulo a multiplicative constant in k^*) such that

$$e_\Upsilon(\pi M) = 0 \text{ for all } M \in Z$$

and

$$e_\Upsilon(x + \pi M) = e_\Upsilon(x) \text{ for all } M \in Z \text{ and } x \in R.$$

We now consider some transcendental questions.

Definition. An element $\theta \in R$ is said to be *σ-algebraic* (over Q) if there exists a non-zero element $M \in Q\{\sigma\}$ such that $M\theta = 0$.

This means that the Q-vector space generated by θ^{σ^n}, $(n \in \mathbf{N})$, is of finite dimension. Of course, an element of R is said to be *σ-transcendental* if it is not σ-algebraic. When σ is the Frobenius then σ-transcendence is equivalent to transcendence over Q.

Theorem 4 [H1]. Assume $\deg \sigma \geq 2$; assume also that $\sigma(t)$ is a power of t, that the degree n of $\varphi(t)$ satisfies $1 < n < d$, and that 0 is neither a pole of a_i $(1 \leq i \leq n)$ nor a zero of a_1. Then $e_\Upsilon(1)$ is o-transcendental.

The proof of Theorem 4 uses an extension of a method due to Wade [W1 – 5].

2. TRANSCENDENCE RESULTS : HISTORICAL SURVEY

We take here $k = \mathbf{F}_q$, $Z = \mathbf{F}_q[t]$, $Q = \mathbf{F}_q(t)$, $R = \mathbf{F}_q((1/t))$, and σ is the Frobenius $\sigma(z) = z^q$. *Carlitz exponential* is

$$e(z) = \sum_{h \geq 0} (-1)^h z^{q^h}/F_h,$$

with

$$F_h = [h][h-1]^q \dots [1]^{q^{h-1}} \text{ and } [h] = t^{q^h} - t,$$

and *Carlitz logarithm is*

$$\log(z) = \sum_{h \geq 0} z^{q^h}/L_h,$$

with $L_h = [h][h-1] \dots [1]$. These series define analytic functions on the completion C of the algebraic closure $\overline{R}$ of R; more precisely $e(z)$ is entire in C, while log is analytic in

$$B = \left\{ z \in C ;\ \deg z < \frac{q}{q-\mathrm{i}} \right\}.$$

The functional equation for $e(z)$ is

$$e(tz) = te(z) - e(z)^q ;$$

its zeroes are $\{M\pi ; M \in \mathbf{F}_q [t]\}$, for some $\pi \in C$ which is uniquely defined modulo $\mathbf{F}_q^*$ (this means that π^{q-1} is well defined). Also we have

$$e(z) = z \prod_{\substack{M \in \mathbf{F}_q[t] \\ M \neq 0}} \left(1 - \frac{z}{M\pi}\right).$$

These functions were introduced by Carlitz in 1935 [*C*]. Six years later, Wade proved the theorem which corresponds to Hermite-Lindemann's theorem on the transcendence of e^α and log α : *if* $\alpha \in C^*$ *is algebraic* (*over* $Q = \mathbf{F}_q(T)$), *then* $e(\alpha)$ is *transcendental* (*over* Q); consequently if $\alpha \in C^* \cap B$ is algebraic, then $\log(\alpha)$ is transcendental. In particular $e(1)$ (which corresponds to the real number 2.71828182...) and π (period of Carlitz exponential, corresponding to 3.14159265...) are both transcendental [W1].

Shortly afterwards [W2], Wade proved the analog of Gel'fond-Schneider's result on the transcendence of α^β : if $\alpha \neq 0$ and $\beta \notin Q$ are algebraic, then $e(\beta \log(\alpha))$ is transcendental. Further transcendence results were also obtained by Wade [W3, 4, 5]; for instance the numbers

$$\sum_{k=1}^{\infty} \frac{1}{[k]}, \quad \sum_{k=1}^{\infty} \frac{1}{x^{r^k}}, \quad \sum_{k=1}^{\infty} \frac{1}{x^{k^s}},$$

for r and s rational integers ≥ 2 with r not a power of p, are all transcendental. Wade's method looks like Euler's proof of the irrationality of the real number e ; let us consider the transcendence proof of the number

$$e(1) = \sum_{h=0}^{\infty} \frac{(-1)^h}{F_h} ;$$

we start by assuming that $e(1)$ is in $\overline{Q}$; then an easy argument shows that there exists a non-zero polynomial

$$A_l X^{q^l} + A_{l-1} X^{q^{l-1}} + \ldots + A_0 \in Z[X]$$

with $e(1)$ as a root. Choose a large rational integer β ; the number

$$F_\beta \sum_{j=0}^{l} \sum_{k=0}^{\infty} \frac{(A_j - 1)^{kq^j}}{F_k^{q^j}}$$

vanishes; Wade decomposes this number in $I_\beta + Q_\beta$, where I_β is in Z while $\deg Q_\beta < 0$. Of course one deduces $I_\beta = Q_\beta = 0$. The most difficult part of the proof is to show that I_β is not congruent to 0 modulo F_β, which yields the desired contradiction.

A quite different method of proof, connected with automata theory, has been given by Allouche [A1] for several of Wade's results. We give here a sketch of proof for the transcendence of

$$\tilde{\pi} = \prod_{j=1}^{\infty} \left(1 - \frac{[j]}{[j+1]}\right) = (t^q - t)^{-1/(q-1)}\, \pi$$

along the method of Allouche.

The main tool is a result of Christol, Kamae, Mendes-France and Rauzy : a formal power series $\sum_{n \geq 0} a_n t^{-n}$ with coefficients a_n in $\mathbf{F}_q$ is algebraic over $\mathbf{F}_q(t)$ if and only if the set of subsequences

$$\{n \to a_{q^k n+r}\ ;\ \ k \geq 0\,, 0 \leq r \leq q^k - 1\}$$

is finite. Let us write

$$\tilde{\pi} = \prod_{j=1}^{\infty} \left(\mathrm{i} - \frac{t^{q^j} - t}{t^{q^{j+1}} - t}\right) = \prod_{j=0}^{\infty} \left(\mathrm{i} - \frac{t^{q^j} - t}{t^{q^{j+1}} - t}\right).$$

Define also $\alpha \in \mathbf{F}_q\,[[1/t]]$ by

$$\alpha = \prod_{j=0}^{\infty} \left(1 - \frac{t^{q^j}}{t^{q^{j+1}}}\right);$$

one readily checks

$$\alpha^q = \prod_{j=0}^{\infty} \left(1 - \frac{t^{q^j}}{t^{q^{j+1}}}\right)^q = \prod_{j=0}^{\infty} \left(1 - \frac{t^{q^{j+1}}}{t^{q^{j+2}}}\right) = \alpha \left(1 - \frac{t}{t^q}\right)^{-1},$$

hence α is algebraic over $\mathbf{F}_q(t)$. On the other hand

$$\frac{\alpha}{\tilde{\pi}} = \prod_{j=0}^{\infty} \left(1 - \frac{1}{t^{q^{j+1}} - 1}\right) = \sum_{n=0}^{\infty} a(n)\, t^{-n}$$

where the sequence $a(n)_{n \geq 0}$ is defined by

$$a(n) = \begin{cases} 0 & \text{if } n \text{ is not of the form } \Sigma_{j \in J}(q^j - 1)\,, \\ (-1)^{\mathrm{Card}\, J} & \text{if there exists a finite set } J \text{ such that } n = \Sigma_{j \in J}(q^j - 1). \end{cases}$$

The desired result that $\tilde{\pi}$ is transcendental is reduced to the fact that the set of sequences

$$\{n \to |\, a(q^k n + r)\,|\ ;\ \ k \geq 0\,, 0 \leq r \leq q^k - 1\}$$

is infinite. Now $|\, a(n)\,|$ is the characteristic function of the set

$$\left\{ n \in \mathrm{N} ; n = \sum_{k=0}^{\infty} \varepsilon_k (q^k - 1),\ \varepsilon_k = 0 \text{ or } 1,\ \varepsilon_k = 0 \text{ for sufficiently large } k \right\}.$$

For each $k \geq 2$, we consider the sequence $(b_k(n))_{n \geq 0}$ which is defined by $b_k(n) = |a(q^k n + q^k - k)|$. One checks that

$$b_k(n) = \begin{cases} 0 & \text{for } 0 \leq n < \dfrac{q^k - 1}{q - 1} - 1, \\ 1 & \text{if } n = \dfrac{q^k - 1}{q - 1} - 1, \end{cases}$$

hence these sequences are pairwise distinct for different values of k. For further details we refer to Allouche's paper [A1].

As noticed by Allouche, the remark that the above number α satisfies $t^q \alpha = (t^q - t) \alpha^q$ extends to the number

$$\gamma = \prod_{j=0}^{\infty} \left(1 - \frac{t^{q^j} - 1}{t^{q^{j+1}} - 1} \right);$$

indeed the following relations hold

$$\gamma = \left(1 - \frac{t - 1}{t^q - 1} \right) \prod_{i=0}^{\infty} \left(1 - \frac{t^{q^{i+1}} - 1}{t^{q^{i+2}} - 1} \right) = \left(\frac{t^q - t}{t^q - 1} \right) \gamma^q,$$

hence $(t^q - 1)\gamma = (t^q - t)\gamma^q$, and γ is algebraic over $\mathbf{F}_q(t)$. In fact $\gamma^{q-1} = \alpha^{q-1}(1 - 1/t)^q$. This corrects a misprint in [Da-H1], which was Allouche's starting point.

In her thesis in 1978 J.M. Geijsel [Gei] developped further the so-called Gel'fond-Schneider method and gave also transcendence results on values of E-functions.

In 1983 Jing Yu started his work on Drinfeld's modules. For simplicity we describe here the situation in a special case (which corresponds to the projective line in place of a more general algebraic curve; see [D], [De-H]).

Let L be a lattice in C (a finitely generated sub-Z-module of C, whose intersection with any *disc* $\{z \in C;\ \deg z < r\}$ is finite). The function

$$e_L(z) = z \prod_{\substack{\alpha \in L \\ \alpha \neq 0}} \left(1 - \frac{z}{\alpha} \right)$$

is entire (the product is uniformly convergent on every disc), it is $\mathbf{F}_q$-additive :

$$e_L(z_1 + z_2) = e_L(z_1) + e_L(z_2)\,, \quad e_L(az) = ae_L(z) \quad \text{for} \quad a \in \mathbb{F}_q\,,$$

and periodic of period lattice L; hence e_L induces an isomorphism of additive groups $C/L \to C$. Since L is a Z-module, the natural Z-module structure of C gives rise via e_L to another Z-module structure on C:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & C & \xrightarrow{e_L} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \Phi_L(a)} & & \\ 0 & \longrightarrow & L & \longrightarrow & C & \xrightarrow{e_L} & C & \longrightarrow & 0 \end{array}$$

for $a \in Z$, where $\Phi_L(a)$ is the endomorphism of C which yields a commutative diagram :

$$e_L(az) = \Phi_L(a)(e_L(z))\,.$$

It is not difficult to check that for each $a \in Z$, the two functions

$$e_L(az) \quad \text{and} \quad \prod_{\gamma \in \frac{1}{a}L/L} (e_L(z) - e_L(\gamma))$$

are proportional (they have the same zeroes). Therefore $\Phi_L(a)$ is a polynomial in σ:

$$\Phi_L(a) = a\,\sigma^0 + a_1\,\sigma + \ldots + a_n\,\sigma^n\,,$$

of degree $n = d \deg a$, where d is the rank of the Z-module L.

Let K be a subfield of C containing Q. The Drinfeld module Φ_L is *defined over* K if $a_i \in K$ for all $a \in Z$ and all i.

Jing Yu stated two conjectures in 1983 [Y1], which he solved completely three years later [Y4] (a quite different solution was also given independently by Dubovitskaia in [Du], using Wade's method of [W1], assuming the rank is $< q$) : *if Φ_L is a Drinfeld module defined over the algebraic closure $\overline{Q}$ of Q in C, then*

— *each non-zero element of L is transcandental,*

— *if $\alpha \neq 0$ is algebraic over Q, then $e_L(\alpha)$ is transcendental.*

In the case where the rank d of L is 1, the Drinfeld module is isomorphic (over $\overline{Q}$) to Carlitz module, and the result follows from Wade's work. In [Y1], Yu proved the second conjecture (and therefore also the first one) in the case $d < q$. In [Y2], he provided a new analog to Gel'fond-Schneider's theorem for a Drinfeld module of rank one : if α_1 and α_2 are non zero and algebraic, and if they have logarithms l_1 and l_2 (which means $e_L(l_i) = \alpha_i$ for $i = 1, 2$) which are

Q-linearly independent, then l_1 and l_2 are also $\overline{Q}$-linearly independent. A version of the *six exponentials theorem* is proved in [Y3] : let Φ_L be a Drinfeld module of rank d which is defined over $\overline{Q}$, let $x_1, \ldots, x_{2d+1}$ be Q-linearly independent elements of C, and let y_1, y_2 be elements of C which are linearly independent over the field K_L of endomorphisms of L. Then at least one of the $4d + 2$ numbers

$$e_L(x_i y_j), \quad (1 \leq i \leq 2d + 1, \quad j = 1, 2)$$

is transcendental.

The main paper of this period [Y4] where Jing Yu provides an analog of the classical criterion of Schneider-Lang for functions of one variable. This enables him to deduce the expected results corresponding to Hermite-Lindemann and Gel'fond-Schneider theorems; for instance he proves the analogs of Schneider's results on the transcendence of complex numbers related to elliptic functions. As a consequence he obtains the transcendence of the values of the Drinfeld modular function j (see [Gel]) at points which are algebraic and not quadratic.

In 1988, there was a special year at the Institute for Advanced Study of Princeton devoted to Drinfeld modules, and there, Jing Yu investigated further the subject by using methods in several variables together with Anderson's theory of motives. His proof [Y5] of the Schneider-Lang criterion in higher dimension involves the same interpolation formulae as in the complex cases (which are due to F. Gross) or the p-adic case (Ph. Robba). This enables him to obtain a first analog of Baker's result, using an idea of Bertrand and Masser. The use of a trace argument restricts the final result to the separable closure of Q; another approach, based on Schneider's method in several variables, enabled recently L. Denis [De] to avoid this restriction for homogeneous linear forms in the case of complex multiplication.

The later works [Y6] and [Y7] of Jing Yu rest on his several dimensional Schneider-Lang criterion, combined with abelian t-modules and Hilbert-Blumenthal-Drinfeld modules; the most interesting feature of these results is that their complex analogs are not yet known : the transcendence theory in finite characteristic is ahead of the complex one. Before we describe these new progresses, we first have to go back half a century ago.

In 1935, Carlitz had also defined zeta values :

$$\zeta(n) = \sum_a \frac{1}{a^n} \in R$$

where n is a positive rational integer, and a runs over the monic elements of Z. These values have also a product expansion

$$\zeta(n) = \prod_p (1 - p^{-n})^{-1},$$

where p runs over the monic irreducible elements of Z. This product also converges for $n = 1$, and $\zeta(1)$ can be considered as the analog of Euler constant

$$\lim_{s \to 1} \left(\zeta \, Q(s) - \frac{1}{s-1} \right)$$

(see [Go1] and [Th4] for connections with gamma functions). Carlitz studied the numbers $\zeta(n)/\pi^n$ for $n \geq 1$, $n \equiv 0 \mod (p-1)$, and proved that they are algebraic : like in Euler's case for the values of the Riemann zeta function at even positive integers, these numbers are essentially quotient of a "Bernouilli number" by a "factorial".

As noticed by Anderson and Thakur [A-Th], Carlitz and Wade knew already enough to prove the transcendence of $\zeta(n)$, (and also of $\zeta(n)/\pi$ if $q \neq 2$), for $n = 1$, and more generally for n a power of p; but this remained considered as an open problem until recently. Anderson and Thakur express $\zeta(n)$ essentially as a coordinate of logarithm of an algebraic point (both at infinity and v-adically), thus reducing the transcendence question to analogues of Hermite-Lindemann, Gel'fond-Schneider-Mahler-Lang results, which were subsequently proved by Jing Yu.

In 1988, Jing Yu [Y6] established the transcendence of $\zeta(n)$ for all $n \geq 1$, and the transcendence of $\zeta(n)/\pi^n$ for all positive $n \not\equiv 0 \mod (p-1)$.

D. Goss has been able to define $\zeta(s)$ for s in a much larger topological space, in particular for s a *negative* integer, and to interpolate these values v-adically for $v \in \operatorname{Spec} R$. Jing Yu also showed that $\zeta_v(n)$ is transcendental for $n \in \mathbb{Z}$, $n > 0$, $n \not\equiv 0 \mod (q-1)$ (this value vanishes if $n \equiv 0 \mod (q-1)$). Furthermore the transcendence results extend to those zeta functions arising from totally real abelian extensions (see [Go1]).

Wade's result on the transcendence of π yields the transcendence of $\zeta(q-1)$; Thakur [Th2] and Damamme and Hellegouarch [Da-H1] had extended this proof to $\zeta(s)$ for $s \leq q^2$, and Damamme [Da2] succeeded to develop Wade's method so as to obtain the transcendence of $\zeta(n)$ for all $n > 0$. This proof is quite different from Jing Yu's one, and should yield further results like the transcendence of values of certain L-functions (see [Da-H2]).

There are a few effective results connected with these qualitative statements. In particular Cherif and de Mathan [Ch1,2], [Ch-M] have surprisingly good estimates for the diophantine approximation of $\zeta(1)$ for $q = 2$, using a method which is inspired by Apéry's proof of the irrationality of $\zeta_Q(3)$. Notice that for $q = 2$, the number $\zeta(n)\pi^{-n}$ is rational for all positive rational integer n, and the conjectures concerning $n \not\equiv 0 \mod (q-1)$ disappear.

Finally, we mention a result of algebraic independence due to A. Thiery [T], namely the analog of the complex theorem due to Chudnovsky which yields the transcendence of $\Gamma(1/4)$ and $\Gamma(1/3)$. It involves the analog of Weierstrass zeta function, introduced by Gekeler [Ge 3,4] and associated with elliptic integrals of second kind.

3. ANDERSON'S t-MOTIVES

In this last lecture we explain briefly what are Anderson's t-motives [A], and we present further transcendence results. Finally we suggest several directions for further researches.

As in section 2, we denote by Z the ring $\mathbf{F}_q[t]$ of polnomials in one variable over the finite field $\mathbf{F}_q$, by Q the field $\mathbf{F}_q(t)$ of rational functions in t, by R its completion $\mathbf{F}_q((1/t))$, and by C the completion of an algebraic closure $\overline{R}$ of R.

We start with a special case which is called the *Carlitz module of dimension* n: this is a homomorphism of k-algebras

$$\Phi: \quad Z \to \operatorname{End}(C^n),$$

which is therefore determined by the image of t:

$$\Phi(t)(x) = \begin{pmatrix} tx_1 + x_2 \\ tx_2 + x_3 \\ \vdots \\ tx_{n-1} + x_n \\ tx_n + x_1^q \end{pmatrix}.$$

In other words $\Phi(t) = tI + N + E\tau$, where

$$\tau\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1^q \\ \vdots \\ x_n^q \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ & & & 1 \\ 0 & & \dots & 0 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}.$$

Hence we replace the ring $C\{\tau\}$, which occured in the one dimensional case, by the ring $C_n\{\tau\}$ of polynomials in τ with coefficients in $M_{n\times n}(Z)$, where

$$\left(\sum_{i=0}^{n} P_i \tau^i\right)\left(\sum_{j=0}^{m} Q_j \tau^j\right) = \sum_{h=0}^{m+n}\left(\sum_{i+j=h} P_i\, Q_j^{\tau^i}\right)\tau^h,$$

where P_i and Q_j are $n \times n$ matrices. Notice that for a $n \times n$ matrix M with entries m_{st}, the entries of the matrix M^{τ^i} are $m_{st}^{q^i}$.

More generally, we define (*) *a t-module of dimension n* as a homomorphism of k-algebras $\Phi : Z \to C_n\{\tau\}$ such that $\Phi(a) = a\tau^0$ for $a \in \mathbb{F}_q$ (where τ^0 is the identity) and

$$\Phi(t) = g_0 + g_1 \tau + \ldots + g_d \tau^d ,$$

with $g_i \in M_{n \times n}(C)$, and g_0 is upper triangular with diagonal $(t, \ldots, t)$.

The exponential map associated to Φ is the unique analytic additive map e_Φ such that

$$e_\Phi(g_0 z) = \Phi(t)\, e_\Phi(z), \quad (z \in C^n),$$

which means that the following diagram is commutative (the left vertical arrow is multiplication by g_0) :

$$\begin{array}{ccc} C^n & \xrightarrow{e_\Phi} & C^n \\ \big\downarrow g_0 & & \big\downarrow \Phi(t) \\ C^n & \xrightarrow{e_\Phi} & C^n \end{array}$$

The Taylor expansion of this function at the origin is

$$e_\Phi(z) = z + \sum_{h=1}^{\infty} \beta_h z^{(h)}, \quad (z \in C^n),$$

for some $\beta_h \in M_{n \times n}(C)$. We shall **assume** that e_Φ is surjective (this property holds at least for Carlitz modules of any dimension). We also assume that the coefficients g_i, $(0 \le i \le d)$ are in $\overline{Q}$; hence the matrices β_h, $(h \ge 1)$ have algebraic entries.

The logarithmic map is a local inverse $\log_\Phi$ to e_Φ :

$$\log_\Phi \circ e_\Phi(z) = e_\Phi \circ \log_\Phi(z) = z,$$

and satisfies the functional equation

$$\log_\Phi(\Phi(t) z) = g_0 \log_\Phi(z).$$

Example. For the Carlitz module of dimension n, one can compute explicitely the components of e_Φ and $\log_\Phi$. The most interesting ones are on one side the first component of the value of e_Φ at a point of which all coordinates but the first one vanish :

(*) The definition varies slightly from one text to another.

$$e_\Phi\begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{i=0}^{\infty} \frac{x^{q^i}}{F_i^n} \\ \vdots \\ \vdots \end{pmatrix},$$

and on the other side the last component of the value of $\log_\Phi$ at a point of which all coordinates but the last one vanish :

$$\log_\Phi\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \\ \displaystyle\sum_{i=0}^{\infty} \frac{x^{q^i}}{L_i^n} \end{pmatrix};$$

as far as the exponential map is concerned, one can notice the similarity with Carlitz functions

$$J_n(t) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{q^{n+i}}}{F_{n+i}F_i^{q^n}}$$

which have ressemblance with classical cylinder (Bessel) functions and whose transcendence properties had been studied by G.M. Geijsel [Gei].

From his criterion of Schneider-Lang in several dimensions (cartesian products), Jing Yu deduces the transcendence of $\sum_{i=0}^{\infty} x^{q^i}/L_i^n$ for $n \geq 1$ and $x \in \overline{Q}$, as well as of $\sum_{i=0}^{\infty} x^{q^i}/F_i^n$. More generally, for a Drinfeld module defined over $\overline{Q}$ as above, when $u \in C^n$ is such that $e_\Phi(u) \in \overline{Q}^n$, each non-zero coordinate of u is transcendental. Recently (Luminy, June 1990), he announced a result which describes all linear dependence relations with coefficients in $\overline{Q}$ between coordinates of such a point u.

Here also the most interesting coordinate of a "logarithm" u (i.e. of a point u such that $e_\Phi(u) \in \overline{Q}^n$) is always the last one. Let us denote by $\mathscr{L}_\Phi \subset C$ the $\overline{Q}$-vector space of so-called "last coordinate logarithms", namely

$$\mathscr{L}_\Phi = \{l \in C;\text{ there exists } \log = (\log_1, \ldots, \log_n) \in C^n \text{ with } e_\Phi(\log) \in \overline{Q}^n \text{ and } \log_n = l\}.$$

Then if $l_1, \ldots, l_n$ are K_Φ-linearly independent elements of $\mathscr{L}_\Phi$, it follows that $1, l_1, \ldots, l_n$ are $\overline{Q}$-linearly independent. Here, K_Φ denotes the ring of endomorphisms of Φ, which is a finite extension of Q.

From the work of Anderson and Thakur [A-Th], it follows that the value $\zeta(n)$ of Carlitz's zeta function is a last coordinate logarithm. Therefore a corollary of Jing Yu's results is the transcendence of $\zeta(n)$ for all $n \geq 1$.

The main point in the proof of Jing Yu is a so-called zero estimate with multiplicities (analog of results in zero characteristic by Masser, Wüstholz and Philippon using Nesterenko's work). Let $\Gamma = Z\gamma_1 + \ldots + Z\gamma_m$ be a finitely generated module in C^n; for $S > 0$ denote by $\Gamma(S)$ the set of $a_1\gamma_1 + \ldots + a_m\gamma_m$ with $a_j \in Z$ satisfying $\deg a_j < S$, $(1 \leq j \leq m)$ (this is a finite set with at most q^{mS} elements). We ask whether there exists a non-zero polynomial P in $C[X_1, \ldots, X_n]$, of degree $\leq D$, which vanishes on $\Gamma(S)$ (*). Obviously if $\Gamma(S)$ has less than $\binom{D+n}{n}$ elements, then indeed such a polynomial exists, as we find its coefficients by solving a system of linear equations where the number of unknowns is less than the number of equations. More generally, if V is a proper $(\neq C^n)$ vector subspace of C^n such that $(\Gamma(S) + V)/V$ has less than $D^{\dim(C^n/V)}$ elements, by considering the projection $C^n \to C^n/V$, one gets a homogeneous linear system of equations which has a non trivial solution. In its simplest version, the zero estimates states that these sufficient conditions for the existence of $P \neq 0$ are also necessary : more precisely there exists a constant $c > 0$ (depending only on the Drinfeld module (C^n, Φ)) such that, if a non-zero polynomial P as above does exist, then there is an integer $s \geq 1$ and a subspace V of C^n stable under the action of t^s, and such that

$$\mathrm{Card}\left(\frac{\Gamma(S-n) + V}{V}\right) \leq c\, D^{\dim(C^n/V)}.$$

This zero estimate is useful for several problems; it is an important tool in connection with Gel'fond's method for algebraic independence (in the complex case this is in fact where the zero estimates appeared for the first time). Here one needs a criterion corresponding to the following complex statement (due to Gel'fond) : if θ is a complex number such that for all sufficiently large N (say $N \geq N_0$) there is a non-zero polynomial $P_N \in \mathbf{Z}[X]$ of degree $\leq N$ and coefficients of absolute values $\leq e^N$ satisfying $|P_N(\theta)| < e^{-6N^2}$, then $P_N(\theta) = 0$ for all $N \geq N_0$ (hence θ is algebraic). The corresponding criterion for Drinfeld modules is given by A. Thiery in [*T*] and has been extended to several variables by *P*. Philippon.

There is another very promising approach to the problem of algebraic independence of logarithms, with the a conjecture due to G. Anderson, the "product principle", which states that the product of two last coordinates logarithms $l_1 \in \mathscr{L}_{\Phi_1}$ and $l_2 \in \mathscr{L}_{\Phi_2}$ is again a last coordinate logarithm $l_3 \in L_{\Phi_3}$ for some

(*) For simplicity we do not consider multiplicities here; the problem is then much aeasier.

suitable Φ_3. In the case of periods (i.e. logarithms of 0), the principle is true and the essential points of the proof can be found in [A].

As suggested by Anderson, "one can expect that by making use of this principle, one should be able to prove results in the spirit of Grothendieck's conjecture in zero characteristic which states that the period matrix of an abelian variety V over $\overline{\mathbf{Q}}$ should generate over $\overline{\mathbf{Q}}$ a field of transcendence degree equal to the dimension of the Mumford Tate group of V".

One can hope that this work of Anderson and Jing Yu will produce the algebraic independence of numbers like $\zeta(3)$ and π in the case of the Cariitz module of dimension 1. In the complex situation, such a statement is still far out of range of the current methods. It may be that the proof of Schanuel's conjecture will achieved earlier in the case of finite characteristic than for the complex field.

The question of diophantine approximation to transcendental numbers connected with Drinfeld modules is almost completely open; apart from Geijsel's [Gei] and Bundschuh's [B] early transcendence measures, and of the work of Cherif and de Mathan, almost nothing is known. Will effective results, like lower bounds for linear combinations of logarithms [De], have the same interest in finite characteristic as they have in the complex or p-adic case?

Another remark is that the known transcendence results for elliptic integrals of the third kind do not have analog for Drinfeld modules.

Of course the main challenge is now to prove in the complex case the statements which correspond to Jing Yu's results, for instance the transcendence of Euler's constant, and of numbers like $\zeta(2n+1)$, $\zeta(2n+1)/\pi^m$ and $\exp(\zeta(2n+1)/\pi^m)$ for integers $n \geq 1$ and $m \in \mathbf{Z}$. Kurokawa conjectures on the other hand for instance that $\zeta(3)/\pi^2$ is the logarithm of an algebraic number, but there is no evidence yet on either side.

References. We give here a short list of references, starting with papers where the main purpose is to establish transcendence results; the second list is a selection of some papers devoted to Drinfeld modules. In June 1991 a conference on Drinfield modules will take place at Ohio State University; the proceedings will be published and this will probably be the best reference for some time on this subject.

REFERENCES ON PAPERS TOTALLY DEVOTED TO TRANSCENDENTAL NUMBERS AND FIELDS OF FINITE CHARACTERISTIC

[WI] WADE, L.I. : *Certain quantities transcendental over GF(p,x)*, Duke Math. J., 8 (1941), 701-720.

[W2] WADE, L.I. : *Certain quantities transcendental over GF(p,x), II*, Duke Math. J., 10 (1943), 587-594.

[W3] WADE, L.I. : *Two types of function field transcendental numbers*, Duke Math. J., 11 (1944), 755-758.

[W4] WADE, L.I. : *Remarks on the Carlitz ψ-functions*, Duke Math. J., 13 (1946), 71-78.

[W5] WADE, L.I. : *Transcendence properties of the Carlitz ψ-functions*, Duke Math. J., 13 (1946), 79-85.

[S] SPENCER, S.M., Jr. : *Transcendental numbers over certain function fields*, Duke Math. J., 19 (1952), 93-105.

[Gei] GEIJSEL, J.M. : Transcendence in fields of positive characteristic, Acad. Proefsch., Amsterdam, 1978.

[B] BUNDSCHUH, P. : *Transzendenzmasse in Körpern formaler Laurentreihen*, J. reine angew. Math., 299/300 (1978), 411-432.

[Y1] YU, JING : *Transcendental numbers arising from Drinfeld modules;* Mathematika, 30 (1983), 61-66.

[Y2] YU, JING : *Transcendence theory over function fields*, Duke Math. J., 52 (1985), 517-527.

[Y3] YU, JING : *A six exponentials theorem in finite characteristic*, Math. Ann., 272 (1985), 91-98.

[Du] DUBOVITSKAYA, N.B. : *Transcendency of analytic parameters associated to elliptic modules*, Mat. Sbornik, 127 (169) $N^o1(5)$ (1985), 131-141.

[Y4] YU, JING : *Transcendence and Drinfeld modules*, Invent. Math., 83 (1986), 507-517.

[Ch1] CHÉRIF, H. : Mesure d'irrationalité de valeurs de la fonction zêta de Carlitz sur $\mathbf{F}_2[T]$, Thèse, Univ. Bordeaux 1, Juin 1987.

[Da1] DAMAMME, G. : *Irrationalité de ζ(s) dans le corps des séries formelle* $\mathbf{F}_q((1/t))$, C.R. Math. Acad. Se. Canada, 9 (1987), 207-212.

[Ch-M] CHÉRIF, H. et DE MATHAN, B. : *Mesure d'irrationalité de la valeur en* 1 *de la fonction zêta de Carlitz, relative à* $\mathbf{F}_2(T)$, C.R. Acad. Sei. Paris, Sér. I, 305 (1987), 761-763.

[Da-H1] DAMAMME, G. et HELLEGOUARCH, Y. : *Propriétés de transcendance de la fonction zêta de Carlitz*, C.R. Acad. Sci. Paris, Sér. I, 307 (1988), 635-637.

[Go1] GOSS, D. : Report on transcendency in the theory of function fields, Number Theory—New-York, 1985-88, Springer Lecture Notes in Math., 1383 (1989), 59-63.

[Y5] YU, JING : *Transcendence and Drinfeld modules: several variables*, Duke Math. J., **58** (1989), 559-575.

[Y6] YU, JING : Transcendence and special zeta values in characteristic p; manuscrit, I.A.S., 1988.

[Y7] YU, JING : On periods and quasi-periods of Drinfeld modules, manuscrit, I.A.S., 1988.

[A1] ALLOUCHE, J.-P. : *Sur la transcendance de la série formelle II*, Sém. Théorie Nombres Bordeaux, **2** (1990), 103-116.

[H1] HELLEGOUARCH, Y. : Généralisation de l'exponentielle de Garlitz et σ-transcendence, manuscrit, 1989.

[Y8] YU, JING : Algebraic independence and Drinfeld modules, manuscrit, 1989.

[Ch2] CHÉRIF, H. : *Mesure d'irrationalité de valeurs de la fonction zêta de Carlitz sur* $\mathbf{F}_q$ *[T]*, C.R. Acad. Sci. Paris, Sér. I, **310** (1990), 23-26.

[Da2] DAMAMME, G. : Transcendance des valeurs de la fonction zêta de Carlitz, Thèse, Caen, Janvier 1990.

[Da-H2] DAMAMME, G. et HELLEGOUARCH, Y. : Transcendence of the values of Carlitz's zeta function by Wade's method, manuscrit, 1990.

[De] DENIS, L. : *Théorème de Baker et modules de Drinfeld*, C.R. Acad. Sci. Paris, Sér. I, **311** (1990), 473-475.

[T] THIERY, A. : Indépendance algébrique des périodes et quasi-périodes d'un module de Drinfeld, manuscrit, 1990.

A SELECTED LIST OF FURTHER REFERENCES ON DRINFELD MODULES

[C] CARLITZ, L. : *On certain functions connected with polynomials in a Galois field*, Duke Math. J., **1** (1935), 137-168.

[D] DRINFEL'D, V.G. : *Elliptic modules*, Mat. Sbornik, 94 (136) (1974), 594-627; trad. angl. : Math. USSR Sbornik, **23** (1974), 561-592.

[Ge1] GEKELER, E.U. : Drinfeld modular curves, Springer Lecture Notes in Math., **1231** (1986).

[A] ANDERSON, G.W. : *t-motives*, Duke Math. J., **53** (1986), 457-502.

[Th1] THAKUR, D.S. : Number fields and function fields (zeta and gamma functions at all prime); N. De Grande-De Kimpe, L. Van Hamme (eds), Proc. conf. on *p*-adic analysis Hengelhoef 1986, 149-157, publi. Vrije Universiteit, Brussels.

[Th2] THAKUR, D.S. : Gamma functions and Gauss sums for function fields and periods of Drinfeld modules, Ph. D. thesis, Harvard, Avril 1987.

[De-H] DELIGNE, P. and HUSEMOLLER, D. : *Survey of Drinfeld modules*, Contemp. Math., **67** (1987), 25-91.

[Th3] THAKUR, D.S. : *Gauss sums for* $\mathbf{F}_q[T]$, Invent. Math., **94** (1988), 105-112.

[Go2] GOSS, D. : *The* Γ *function in the arithmetic of function fields*, Duke Math. J., 56 (1988), 163-191.

[Ge2] GEKELER, E.U. : *On the coefficients of Drinfeid modular forms*, Invent. Math., 93 (1988), 667-700.

[H2] HELLEGOUARCH, Y. : *Notions de base pour l'arithmétique de* $\mathbf{F}_q(1/t)$, Can. J. Math., **40** (1988), 817-832.

[A-Th] ANDERSON, G. W. and THAKUR, D. : *Tensor powers of the Carlitz module and zeta values*, Annals of Math., **132** (1990), 159-191.

[Ge3] GEKELER, E.U. : *Quasi-periodic functions and modular forms*, Comp. Math., 69 (1989), 277-293.

[Ge4] GEKELER, E.U. : *On the de Rham isomorphisms for Drinfeid modules*, J. reine angew. Math., **401** (1989), 188-208.

[Go3] GOSS, D. : *Fourier series, measures and divided power series in the theory of function fields*, K-Theory, **1** (1989), 533-555.

[Th4] THAKUR, D.S. : *Gamma functions for function fields and Drinfeid modules*, Annals of Math., to appear.

[H3] HELLEGOUARCH, Y. : *Calcul différentiel galoisien*, prépublication N° 42, Univ. Caen, 1989.

[H4] HELLEGOUARCH, Y. : *Modules de Drinfeid généralisés*, in **Approximations Diophantiennes et Nombres Trancendants**, Luminy 1990, à paraître.

UNIVERSITE P. ET M. CURIE (PARIS VI)
C.N.R.S. "PROBLEMES DIOPHANTIENS"
INSTITUT HENRI POINCARE
11, RUE P. ET M. CURIE
75231 PARIS CEDEX 05