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ON p-ADIC U_m-NUMBERS

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In this paper it is shown that integral combination with p-adic algebraic coefficients of some certain p-adic Liouville numbers belong to the Mahler subclass U_m in the Hensel's field $\mathbf{Q}_p$ of p-adic numbers, where m is the degree of the algebraic number field determined by these coefficients. Thus we have carried the results in [4] to the the p-adic case.

In the following p is a fixed prime of $\mathbf{Q}$ and $|\ldots|_p$ denotes the p-adic valuation.

Definition[1]. Let ξ be a p-adic number in $\mathbf{Q}_p$ and $m \geq 1$ an integer. The number ξ is called p-adic U_m-number if for every $w > 0$ there are infinitely many algebraic numbers γ of degree m with

$$0 < |\xi - \gamma|_p < H(\gamma)^{-w}$$

and if there exist constants $C, K > 0$ depending only on ξ and m such that the relation

$$|\xi - \beta|_p > C\,H(\beta)^{-K}$$

holds for every algebraic number β in $\mathbf{Q}_p$ which has degree less than m.

Lemma I. Let $P(x) = a_0 + a_1 x + \ldots + a_k x^k$ be a polynomial of degree k with integral coefficients and α be a p-adic algebraic number of degree M with $P(\alpha) \neq 0$. Then the relation

$$|P(\alpha)|_p \geq \frac{p^{(M-1)t}}{(M+k)!\,H(P)^M H(\alpha)^k} \tag{1}$$

holds, where $|\alpha|_p = p^{-h}$, $t = \min(0, h)$, and $H(P)$, $H(\alpha)$ are the height of $P(x)$ and the height of the minimal polynomial of α respectively (K. Mahler [2]).

[1] We note that we have, in fact, defined a p-adic U_m^*-number in [1] instead of p-adic Mahler U_m-number. However, it is known that they are the same (see [1], [2]).

Now using (1) we give a lower bound for $|\alpha - \beta|_p$ where β is an arbitrary p-adic algebraic number of degree $k < M$. If $|\beta|_p \neq |\alpha|_p$, then we have $|\alpha - \beta|_p > p^{-|h|}$. Hence we may assume that $|\alpha - \beta|_p \leq 1$ and $|\beta|_p = p^h$.

Let $P(x)$ be the minimal polynomial of β. Then

$$0 \neq P(\alpha) = P(\beta) + (\alpha - \beta)\, P'(\beta) + (\alpha - \beta)^2 \frac{P''(\beta)}{2!} + \ldots$$

and so

$$0 < |P(\alpha)|_p = |\alpha - \beta|_p \left| P'(\beta) + (\alpha - \beta) \frac{P''(\beta)}{2!} + \ldots \right|_p .$$

Thus using $|P^{(j)}(\beta)|_p \leq p^{M|h|}$ and $\left|\frac{1}{j!}\right|_p \leq p^M \ (1 \leq j < M)$ we see that the second factor on the right side of the above equality $\leq p^{M(|h|+1)}$. Hence using this in (1) we get

$$|\alpha - \beta|_p \geq c_0 H(\alpha)^{-M+1}\, H(\beta)^{-M}, \tag{2}$$

where $c_0 = p^{(M-1)t - M(|h|+1)}\, (2M!)^{-1}$ is a constant depending only on α.

Lemma II. Let $\alpha_1, \ldots, \alpha_k$ $(k \geq 1)$ be algebraic numbers in $\mathbf{Q}_p$ with $[\mathbf{Q}(\alpha_1, \ldots, \alpha_k) : \mathbf{Q}] = g$ and let $F(y, x_1, x_2, \ldots, x_k)$ be a polynomial with integral coefficients, whose degree in y is at least one. If η is an algebraic number such that $F(\eta, \alpha_1, \ldots, \alpha_k) = 0$, then the degree of $\eta \leq dg$ and

$$h_\eta \leq 3^{2dg + (l_1 + \ldots + l_k)g}\, H^g\, h_{\alpha_1}^{l_1 g} \ldots h_{\alpha_k}^{l_k g},$$

where h_η is the height of η, h_{α_i} is the height of α_i $(i = 1, \ldots, k)$, H is the maximum of the absolute values of the coefficients of F, l_i is the degree of F in x_i $(i = 1, \ldots, k)$ and d is the degree of F in y (O.Ş. İçen [5]).

Lemma III. Let $\alpha_0, \alpha_1, \ldots, \alpha_k$ $(k \geq 1,\ \alpha_i \neq 0,\ i = 0, 1, \ldots, k)$ be algebraic numbers in $\mathbf{Q}_p$ with $[\mathbf{Q}(\alpha_0, \ldots, \alpha_k) : \mathbf{Q}] = m > 1$ and let $\{u_n^{(1)}\}, \{u_n^{(2)}\}, \ldots, \{u_n^{(k)}\}$ be sequences of positive integers with

$$\lim_{n \to \infty} u_n^{(i)} = \infty \quad (i = 1, \ldots, k), \tag{3a}$$

$$\lim_{n \to \infty} \frac{\log u_n^{(i+1)}}{\log u_n^{(i)}} = \infty \quad (i = 1, \ldots, k-1). \tag{3b}$$

Then there exists a positive integer N such that if $n > N$, the degree of the algebraic number $\gamma_n = \alpha_0 + \sum_{i=1}^{k} u_n^{(i)} \alpha_i$ is m and

$$\lim_{n \to \infty} H(\gamma_n) = \infty . \tag{4}$$

The proof is the same as in the Lemma III in [4]. Now applying the LeVeque's idea in [6] to p-adic case we have the

Theorem I. Let $\alpha_0, \alpha_1, \ldots, \alpha_k$ ($\alpha_i \neq 0, i = 0, 1, \ldots, k, k \geq 1$) be algebraic numbers in $\mathbf{Q}_p$ with $[\mathbf{Q}(\alpha_0, \ldots, \alpha_k) : \mathbf{Q}] = m > 1$ and let $\xi_1, \xi_2, \ldots, \xi_k$ be p-adic Liouville numbers in the canonical forms

$$\xi_i = a_0^{(i)} + a_1^{(i)} p^{u_1^{(i)}} + \ldots + a_n^{(i)} p^{u_n^{(i)}} + \ldots \tag{5}$$

$$(u_\nu^{(i)} > 0,\ u_{\nu+1}^{(i)} > u_\nu^{(i)},\ 1 \leq a \leq p-1,\ 1 \leq i \leq k, \nu = 1, 2, \ldots),$$

where

$$\lim_{n\to\infty} \frac{u_{n+1}^{(i)}}{u_n^{(i)}} = \infty \quad (i = 1, \ldots, k).$$

Next assume that monotonic union sequence s_n (consisting of all integers $s = u_j^{(i)}$ for i, j, arranged by size) satisfies that

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \infty. \tag{6}$$

Then the p-adic number $\gamma = \alpha_0 + \alpha_1 \xi_1 + \ldots + \alpha_k \xi_k$ is a p-adic U_m-number.

Proof. Let N_0 be a positive integer with $N_0 \geq \max_{i=1}^{k} (u_1^{(i)})$ and n_0 be an integer such that if $n > n_0$ then $s_n > N_0$. For $n > n_0$ we define integers $r_i(n)$ and $\rho_n^{(i)}$ by

$$u_{r_i(n)}^{(i)} = \max_j \{u_j^{(i)} \mid u_j^{(i)} \leq s_n\} \quad (i = 1, \ldots, k),$$

$$\rho_n^{(i)} = a_0^{(i)} + a_1 p^{u_1^{(1)}} + a_2 p^{u_2^{(2)}} + \ldots + a_{r_i(n)} p^{u_{r_i(n)}^{(i)}} \quad (i = 1, \ldots, k) \tag{7}$$

and algebraic numbers

$$\gamma_n = \alpha_0 + \alpha_1 \rho_n^{(1)} + \alpha_2 \rho_n^{(2)} + \ldots + \alpha_k \rho_n^{(k)} \quad (n > n_0). \tag{8}$$

Now to prowe that $\gamma \in \bigcup_{j=1}^{m} U_j$ we shall approximate γ by γ_n ($n > n_0$). First we have

$$|\gamma - \gamma_n|_p \leq \max_{i=1}^{k} \{|\alpha_i|_p\} \max_{i=1}^{k} \{|\xi_i - \rho_n^{(i)}|_p\}. \tag{9}$$

On the other hand it follows from the definitions of ξ_i and $\rho_n^{(i)}$ that

$$|\xi_i - \rho_n^{(i)}|_p \leq p^{-s_{n+1}} \quad (n > n_0,\ i = 1, \ldots, k).$$

Thus putting $c_1 = \max_{i=1}^{k} (|\alpha_i|_p)$ and using the above inequality in (9)

$$|\gamma - \gamma_n|_p \leq c_1 p^{-s_{n+1}}. \tag{10}$$

Next applying Lemma II to $\gamma_n, \alpha_0, \ldots, \alpha_k$ in (8) we obtain

$$H(\gamma_n) \leq c_2 p^{m.s_n},$$

where c_2 is a constant depending only on $p, m, k, \alpha_0, \ldots, \alpha_k$. Since $s_n \to \infty$ as $n \to \infty$, there is a positive integer n_1 such that if $n > n_1$ then

$$H(\gamma_n) \leq p^{2ms_n}. \tag{11}$$

Finally using this in (10) we get

$$|\gamma - \gamma_n|_p \leq c_1 \; H(\gamma_n)^{-(s_{n+1}/2ms_n)} \qquad (n > \max(n_0, n_1)), \tag{12}$$

which gives us that $\gamma \in \bigcup_{j=1}^{m} U_j$ by (6). Now to complete the proof we must show that $\gamma \notin U_j$ $(j = 1, \ldots, m-1)$. It can be seen from (5) and (6) that γ_n satisfies all conditions in Lemma III. Hence there is a positive integer n_2 such that if $n > n_2$ then degree of $\gamma_n = m$.

Let β be a p-adic algebraic number of degree $< m$. Then we can apply Lemma I to β, γ_n $(n > n_2)$ and so we obtain

$$|\gamma_n - \beta|_p \geq c_3 \; H(\gamma_n)^{-m+1} H(\beta)^{-m}$$

or using (11) in the above inequality

$$|\gamma_n - \beta|_p \geq c_3 p^{\,2m(m-1)s_n} H(\beta)^{-m}, \;\; n > \max_{i=0}^{2} \{n_i\}, \tag{13}$$

where c_3 is a positive constant depending only on p, m, α_i, ξ_i $(1 \leq i \leq k)$. Set $t(m) = 2m^2 - m + 1$ and $r(m) = 2m(m-1)\,t(m) + m + 1$. Then there is an integer n_3 such that if $n > n_3$ then $\frac{s_{n+1}}{s_n} > r(m)$. On the other hand for every $H(\beta) > \max\left(\frac{c_1}{c_3}, p^{s_{\max_{i=0}^{3}(n_i)}}\right)$ there is an integer ν such that

$$p^{s_\nu} < H(\beta) \leq p^{s_{\nu+1}}. \tag{14}$$

Now we have two cases in (14) as following:

Case I. Let $p^{s_\nu} < H(\beta) \leq p^{s_{\nu+1}/t(m)}$. Then using the first and second part of this inequality in $(13)_{n=\nu}$ and in $(10)_{n=\nu}$ respectively we obtain

$$|\gamma_\nu - \beta|_p \geq c_3 H(\beta)^{-r(m)+1}, \; |\gamma - \gamma_n|_p \leq c_1 H(\beta)^{-r(m)}$$

that is

$$|\gamma_\nu - \beta|_p > |\gamma - \gamma_n|_p$$

and so

$$|\gamma - \beta|_p = \max(|\gamma_\nu - \beta|_p, |\gamma - \gamma_\nu|_p) \geq c_3 H(\beta)^{-r(m)+1}.$$

Case II. If $p^{s_{\nu+1}/r(m)} < H(\beta) \leq p^{s_{\nu+1}}$ then writing (10) and (13) for $n = \nu + 1$ and using the above inequality we see that

$$|\gamma_{\nu+1} - \beta|_p \geq c_3 H(\beta)^{-r(m)+1}, \; |\gamma - \gamma_{\nu+1}|_p \leq c_1 H(\beta)^{-s_{\nu+2}/s+1}$$

or

$$|\gamma_{\nu+1} - \beta|_p > |\gamma - \gamma_{\nu+1}|_p.$$

Finally this inequality gives us that $|\gamma - \beta|_p \geq C_3 H(\beta)^{-r(m)+1}$ and this completes the proof.

Example. Let $k > 1$ be an integer. Then p-adic Liouville numbers

$$\xi_{i+1} = 1 + p^{(k+i)!} + p^{(2k+i)!} + \ldots + p^{(nk+i)!} + \ldots \quad (i = 0, 1, \ldots, k-1),$$

satisfy all conditions in Theorem I. So if $\alpha_0, \alpha_1, \ldots, \alpha_k$ are p-adic algebraic numbers with $[\mathbf{Q}(\alpha_0, \ldots, \alpha_k) : \mathbf{Q}] = m$ then $\alpha_0 + \alpha_1 \xi_1 + \ldots + \alpha_k \xi_k \in U_m$.

Note that it can be easily seen from the proof of Theorem I that it is sufficient to suppose that $\liminf_{n\to\infty} \frac{s_{n+1}}{s_n} > r(m)$ and $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} = \infty$ instead of stronger assumption (6).

We have a generalization of Theorem I as

Theorem II. Let $\alpha_0, \alpha_1, \ldots, \alpha_k$ $(k \geq 1)$ be p-adic algebraic numbers in $\mathbf{Q}_p$ with $[\mathbf{Q}(\alpha_0, \ldots, \alpha_k) : \mathbf{Q}] = m$ and $\xi_1, \xi_2, \ldots, \xi_k$ be p-adic Liouville numbers in the canonical forms

$$\xi_i = a_0^{(i)} + a_1^{(i)} p^{u_1^{(i)}} + \ldots + a_n^{(i)} p^{u_n^{(i)}} + \ldots$$

$$(u_{j+1}^{(i)} > u_j^{(i)} > 0, \; 1 \leq a_j \leq p-1, \; i = 1, \ldots, k, \; j = 1, 2, \ldots),$$

and suppose that the sequence $\{u_n^{(i)}\}$ has a subsequence $\{u_{\nu_n^{(i)}}^{(i)}\}$ verifying the conditions

a) $$\lim_{n\to\infty} \frac{u^{(i)}_{\nu_n^{(i)}+1}}{u^{(i)}_{\nu_n^{(i)}}} = \infty \quad (i = 1, \ldots, k),$$

b) $$\lim_{n\to\infty} \frac{u^{(i)}_{\nu_{n+1}^{(i)}+1}}{u^{(i)}_{\nu_n^{(i)}+1}} < \infty \quad (i = 1, \ldots, k).$$

Further suppose that the monotonic union sequence s_n (consisting of all integers $s = u^{(i)}_{v^{(i)}_j}$ for i, j, arranged by size) satisfies the condition $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \infty$, then the p-adic number $\gamma = \alpha_0 + \alpha_1 \xi_1 + \dots + \alpha_k \xi_k$ is a p-adic U_m-number.

We define integers $r_i(n)$ and $\rho^{(i)}_n$ as following:

$$u^{(i)}_{r_i(n)} = \max_t \{u^{(i)}_{v^{(i)}_t} \mid u^{(i)}_{v^{(i)}_t} \leq s_n\},\ \rho^{(i)}_n = \sum_{j=0}^{r_i(n)} a^{(i)}_j\, p^{u^{(i)}_j} \quad (i = 1, \dots, k,\ n = \mathrm{i}, \dots).$$

Then we approximate γ by $\gamma_n = \alpha_0 + \alpha_1 \rho^{(1)}_n + \alpha_2 \rho^{(2)}_n + \dots + \alpha_k \rho^{(k)}_n$. The proof, which we shall omit, can be conducted by using a combination of the arguments used in the proof of Th. I. Finally we have the

Corollary I. Let ξ be a p-adic Liouville number in the canonical form $\xi = a_0 + a_1 p^{u_1} + \dots + a_n p^{u_n} + \dots$ $(u_{i+1} > u_i > 0,\ 1 \leq a_i \leq p-1,\ i = 1, \dots)$, and suppose that the sequence $\{u_n\}$ has a subsequence $\{u_{v_n}\}$ verifying the conditions

$$\lim_{n\to\infty} \frac{u_{v_{n+1}}}{u_{v_n}} = \infty,\quad \limsup_{n\to\infty} \frac{u_{v_{n+1}}}{u_{v_n+1}} < \infty.$$

Then p-adic Liouville number ξ can be represented by the sum and product of two p-adic U_m $(m = 1, \dots)$ numbers.

Proof. First note that by Lemma I in [1] we know that for every integer $m > 1$, there is an algebraic number α of degree m in $\mathbf{Q}_p$. Now if $m = 1$ then there is nothing to prove since $\xi = \frac{\xi}{2} + \frac{\xi}{2} = \xi^2 . \xi^{-1}$ and $\frac{\xi}{2}, \xi^2, \xi^{-1}$ are p-adic Liouville numbers. Let $m > 1$ and $\alpha \in \mathbf{Q}_p$ with $\deg \alpha = m$. Then

$$\xi = \left(\frac{\xi}{2} + \alpha\right) + \left(\frac{\xi}{2} - \alpha\right),\quad \xi = (\xi^2 \alpha)(\alpha \xi)^{-1}.$$

One can show that ξ^2 and ξ^{-1} also satisfy the conditions in the corollary. Thus by Theorem II we see that

$$\frac{\xi}{2} \mp \alpha,\ \xi^2 \alpha,\ (\alpha \xi)^{-1} \in U_m$$

and this completes the proof.

Corollary II. Every algebraic number α in $\mathbf{Q}_p$ of degree $m > 1$ can be represented by the sum and product of two p-adic U_{mk} $(k = 1, \dots)$ numbers.

Proof. Let $\alpha \in \mathbf{Q}_p$ with $\deg \alpha = m > 1$. Then for $k \geq 1$, there is an algebraic number β in $\mathbf{Q}_p$ with $\deg \beta = k$ such that

$$\deg\left(\frac{\alpha}{2} \mp \beta\right) = \deg \frac{\alpha+1}{\alpha}\,\beta = \deg \frac{\beta}{\alpha+1} = km.$$

Let ξ be a p-adic Liouville number satisfying the conditions in Corollary I. Then we have

$$\alpha = \left(\frac{\alpha}{2} - \beta + \xi\right) + \left(\frac{\alpha}{2} + \beta - \xi\right), \; \alpha = \left(\frac{\alpha+1}{\beta}\,\alpha\,\xi\right)\left(\frac{\beta}{(\alpha+1)\,\xi}\right).$$

But it follows from Th. II that

$$\frac{\alpha}{2} \mp \beta \pm \xi \in U_{mk}\,, \; \frac{\alpha+1}{\beta}\,\alpha\,\xi\,, \; \frac{\beta}{(\alpha+1)\,\xi} \in U_{mk}$$

and this completes the proof.

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ÖZET

Bu çalışmada bazı koşulları gerçekleyen Liouville sayılarının cebirsel katsayılı tam kombinezonları incelenerek, bunların Q_p cismindeki Mahler'in U_m alt sınıfına ait olduğu gösterilmektedir.