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ON THE SYMMETRY DIGRAPH SYM (Γ)

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In this paper we study some properties of SYM (Γ) for some digraphs Γ and we prove some theorems related to Aut (Γ) and to SYM (Γ). A theorem on group graphs is also proved.

DEFINITIONS

1. A directed graph (digraph) Γ consists of two disjoint sets Γ_V and Γ_E called the set of vertices and the set of edges respectively, and two functions $\sigma, \tau : \Gamma_E \longrightarrow \Gamma_V$, called the source and target maps respectively.

It is sometimes convenient to distinguish between those edges where the source and target maps coincide and those where they differ. A loop is an edge e such that $\sigma\ e = \tau\ e$, and a link is an edge e such that $\sigma\ e \neq \tau\ e$.

2. A morphism of digraphs $\alpha : \Gamma \longrightarrow \Delta$ is a pair of functions $(\alpha_V : \Gamma_V \longrightarrow \Delta_V,$ $\alpha : \Gamma_E \longrightarrow \Delta_E)$ such that $\sigma\ \alpha_E\ e = \alpha_V\ \sigma\ e$ and $\tau\ \alpha_E\ e = \alpha_V\ \tau\ e$, for each $e \in \Gamma_E$.

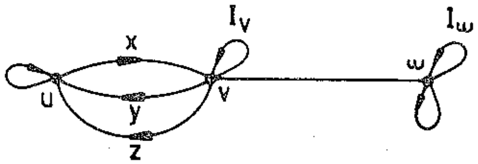
Note that we abuse notation by using the same symbols σ, τ for the source and target maps in any digraph, this should not result in any confusion.

3. A morphism $\alpha : \Gamma \longrightarrow \Gamma$ is called invertible if there exists a morphism $\alpha^{-1} : \Gamma \longrightarrow \Gamma$ such that $\alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = I_\Gamma$ where I_Γ the identity morphism on Γ.

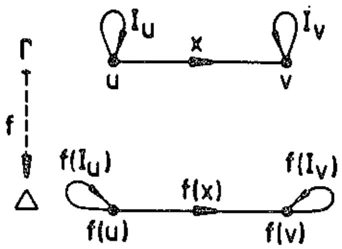
4. The automorphism group of a graph Γ is the set of all invertible morphisms $\alpha^{-1} : \Gamma \longrightarrow \Gamma$, and it is denoted by Aut (Γ).

Let I_u denote the loop at the vertex u of Γ and $\partial : \Gamma_V \longrightarrow \{I_v : v \in \Gamma_V\}$ defined by $\partial(v) = I_v$, then we can think of the vertices of Γ as loops, it is easily shown that $\sigma(I_u) = \tau(I_u) = I_u$, $\tau^2 = \sigma \circ \tau = \tau$ and $\tau \circ \sigma = \sigma^2$, also $\sigma \circ \partial = \partial$ and $\tau \circ \partial = \partial$. We can simply say that a morphism $\alpha : \Gamma \longrightarrow \Delta$ is any map which preserves σ, τ and ∂. The vertices of a digraph will be represented in diagrams by dots and edges by arrows from the source vertex to the target vertex.

Example 1. The following diagram represents a digraph :



Example 2. The following diagram represents a morphism f between the digraphs Γ and Δ:



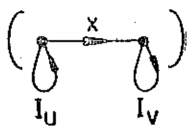
5. For convenience, we now group together a number of definitions which will be needed later.

Let $\Gamma(u, v) = \{x \in \Gamma : \sigma\, x = u,\ \tau\, x = v\}$, we shall refer to the sets $\Gamma(u, v)$ where $u, v \in \Gamma$ as the edge sets of Γ (even though an "edge set" $\Gamma(v, v)$ will contain the vertex v, and might contain nothing else). A multiple edge set is one containing more than one element.

We say that a digraph Γ is complete if there is a bijection $\Gamma(u, v) \cong \cong \Gamma(u', v')$ for each two pairs of (not necessarily distinct) vertices u, v and u', v' in Γ. In particular if $|\Gamma(u, v)| = n$ for any two vertices u, v in Γ we shall say that Γ is n-complete.

THE STRUCTURE OF SYM (Γ)

The endomorphism digraph END(Γ) of a digraph Γ consists of all triples (f, β, γ) such that, f, β, γ are endofunctions of Γ, β and γ are morphisms and $\sigma f = \beta\, \sigma$, $\tau f = \gamma\, \tau$. These conditions give for each $x \in \Gamma \equiv$



a diagram **D**

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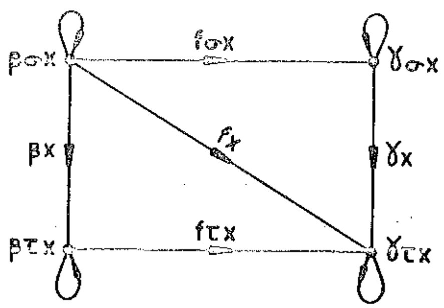
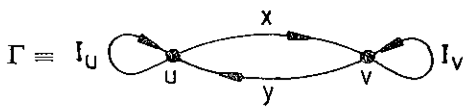


The source and target functions take (f, β, γ) to (β, β, β) and (γ, γ, γ) respectively. So the vertices of END (Γ) are essentially the endomorphisms of Γ. The monoid multiplication on END (Γ) is given by pointwise composition of functions : $(f, \beta, \gamma)\ (f', \beta', \gamma') = (ff', \beta\beta', \gamma\gamma')$.

The symmetry digraph SYM (Γ) of Γ consists of the invertible elements of END (Γ), in other words the triples (f, β, γ) such that f, β, γ are permutations of Γ, β and γ are morphisms and $\sigma \circ f = \beta \circ \sigma$, $\tau \circ f = \gamma \circ \tau$ (note that we use the term "permutation" for a one - one correspondence of a set to itself, whether the set is finite or infinite). Thus the automorphism group Aut (Γ) forms the set of vertices of SYM (Γ).

Example 3. Let Γ be a digraph whose diagram is given below. Then $\text{Aut}(\Gamma) = (i, j)$ where i is the identity automorphism of Γ, and j is defined

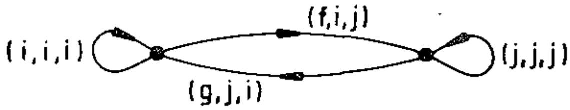


by $j(I_u) = I_u$, $j(I_v) = I_v$, $j(x) = y$ and $j(y) = x$.

The symmetry digraph SYM (Γ) has only two vertices $\{i, j\}$ and it has four edges that can be calculated as follows : Let Z represent the edges of Γ

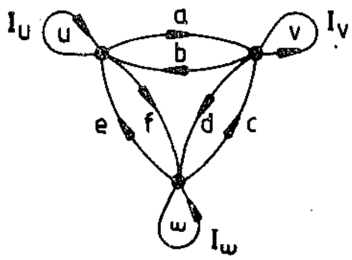
edges of (SYM (Γ))	edges of Γ	$(\sigma \circ f)$	$(\tau \circ f)$	$F(Z)$
$(f, i, i) \equiv (i, i, i)$	I_u	u	u	I_u
	I_v	v	v	I_v
	x	u	v	x
	y	v	u	y
(f, i, j)	I	u	u	I
	I	v	u	y
	x	u	u	I
	y	v	v	I
(g, j, i)	I	v	u	y
	I	u	v	x
	x	v	v	I
	y	u	u	I
$(g, j, j) = (j, j, j)$	I	v	v	I
	I	u	u	I
	x	v	u	y
	y	u	v	x

so $\mathrm{SYM}(\Gamma)=\{(i,i,i),(f,i,j),(g,j,i),(j,j,j)\}$, and it can be represented by the following diagram :



Example 4. Consider the 1-complete digraph Γ (exactly one directed edge from any vertex to any other) on three vertices,

$\Gamma \cong$



then we get :

$\mathrm{Aut}(\Gamma)=\{\epsilon=i,\ \beta=(u\,v\,w),\ \beta^2=(u\,w\,v),\ \gamma=(u\,v),\ \beta\gamma=(u\,w),\ \beta^2\gamma=(v\,w)\}$, and

$\mathrm{SYM}(\Gamma)=\{(\alpha,\epsilon,\epsilon)=$ identity, $(\alpha,\epsilon,\beta)=(I_u\,a\,f)(I_v\,c\,b)(I_w\,e\,d)$,

$(\alpha,\beta,\epsilon)=(I_u\,b\,e)(I_v\,d\,a)(I_w\,e\,d)$,
$(\alpha,\beta^2,\epsilon)=(I_u\,e\,b)(I_v\,a\,d)(I_w\,c\,f)$,
$(\alpha,\epsilon,\beta\gamma)=(I_u\,f)(I_w\,e)(b\,c)$,
$(\alpha,\epsilon,\beta^2\gamma)=(I_v\,c)(I_w\,d)(a\,f)$,
$(\alpha,\beta,\beta)=(I_u\,I_v\,I_w)(a\,c\,e)(b\,d\,f)$,
$(\alpha,\beta,\beta^2)=(I_u\,c\,d)(I_v\,e\,f)(I_w\,a\,b)$,
$(\alpha,\beta,\gamma)=(I_u\,I_v\,e\,a\,b\,d)(I_w\,f\,c)$,
$(\alpha,\beta,\beta\gamma)=(I_u\,c\,e\,f\,b\,I_w)(I_v\,d\,a)$,
$(\alpha,\beta,\beta^2\gamma)=(I_u\,b\,c)(I_v\,I_w\,a\,c\,d\,f)$,
$(\alpha,\beta^2,\gamma)=(I_u\,d\,b\,a\,e\,I_v)(I_w\,c\,f)$,
$(\alpha,\beta^2,\beta\gamma)=(I_u\,I_w\,b\,f\,e\,c)(I_v\,a\,d)$,
$(\alpha,\beta^2,\beta^2\gamma)=(I_u\,e\,b)(I_v\,f\,d\,c\,a\,I_w)$,
$(\alpha,\gamma,\gamma)=(I_u\,I_v)(a\,b)(c\,f)(d\,e)$,
$(\alpha,\beta\gamma,\beta\gamma)=(I_u\,I_w)(a\,d)(b\,c)(e\,f)$,
$(\alpha,\beta^2\gamma,\beta^2\gamma)=(I_v\,I_w)(a\,f)(b\,e)(c\,d)$,
$(\alpha,\gamma,\beta\gamma)=(I_u\,c)(I_v\,a)(I_w\,e)(b\,f)$,
$(\alpha,\gamma,\beta^2\gamma)=(I_u\,b)(I_v\,f)(I_w\,d)(a\,c)$,
$(\alpha,\beta\gamma,\beta^2\gamma)=(I_u\,e)(I_v\,c)(I_w\,a)(d\,f)$,
$(\alpha,\epsilon,\beta^2)=(I_u\,f\,a)(I_v\,b\,c)(I_w\,d\,e)$,
$(\alpha,\epsilon,\gamma)=(I_u\,a)(I_v\,b)(d\,e)$,
$(\alpha,\gamma,\epsilon)=(I_u\,b)(I_v\,a)(c\,f)$,
$(\alpha,\beta\gamma,\epsilon)=(I_u\,e)(I_w\,f)(a\,d)$,
$(\alpha,\beta^2\gamma,\epsilon)=(I_v\,d)(I_w\,c)(b\,e)$,
$(\alpha,\beta^2,\beta^2)=(I_u\,I_w\,I_v)(a\,e\,c)(b\,f\,d)$,
$(\alpha,\beta^2,\beta)=(I_u\,d\,c)(I_v\,f\,e)(I_w\,b\,a)$,
$(\alpha,\gamma,\beta)=(I_u\,I_v\,f\,b\,a\,c)(I_w\,e\,d)$,
$(\alpha,\beta\gamma,\beta)=(I_u\,d\,f\,e\,a\,I_w)(I_v\,c\,b)$,
$(\alpha,\beta^2\gamma,\beta)=(I_u\,a\,f)(I_v\,I_w\,b\,d\,c\,e)$,
$(\alpha,\gamma,\beta^2)=(I_u\,c\,a\,b\,f\,I_v)(I_w\,d\,e)$,
$(\alpha,\beta\gamma,\beta^2)=(I_u\,I_w\,a\,e\,f\,d)(I_v\,b\,c)$,
$(\alpha,\beta^2\gamma,\beta^2)=(I_u\,f\,a)(I_v\,e\,c\,d\,b\,I_w)$,
$(\alpha,\beta\gamma,\gamma)=(I_u\,d)(I_v\,b)(I_w\,f)(a\,e)$,
$(\alpha,\beta^2\gamma,\gamma)=(I_u\,a)(I_v\,e)(I_w\,c)(b\,d)$,
$(\alpha,\beta^2\gamma,\beta\gamma)=(I_u\,f)(I_v\,d)(I_w\,b)(c\,e)\}$.

We will study this example later on.

Theorem [1]. If Γ is 1-complete digraph on n-vertices, then $\mathrm{Aut}(\Gamma)$ is isomorphic to the symmetric group S_n.

Theorem. If Γ is 1-complete digraph on n-vertices, then SYM (Γ) is 1-complete on $n!$ vertices.

Proof. Let β and γ belong to Aut (Γ), i.e. β and γ are two vertices in SYM(Γ). Γhen there is an edge (f, β, γ) where f is a bijective function on Γ, if it is not then there are two distinct edges x and y in Γ_E such that $\beta \circ o(x)=\beta \circ \sigma(y)$ and $\gamma \circ \tau(x)=\gamma \circ \tau(y)$, which means that $\sigma(x)=o(y)$ and $\tau(x)=\tau(y)$ but since Γ is a complete digraph, then the two directed edges x and y coincide and this contradicts that they are distinct from this and from the above theorem, we prove that SYM (Γ) is complete on $n!$ vertices.

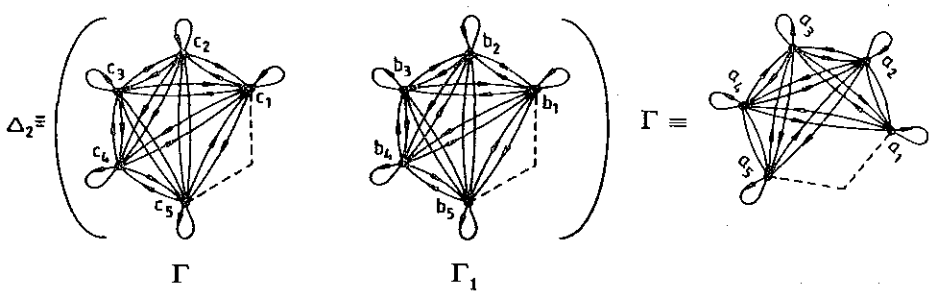
Theorem. If Γ is a discrete digraph (i.e. $\Gamma_E=\{I_v, v \in \Gamma_v\}$) then SYM (Γ) is discrete on $n!$ vertices.

Proof. It is clear that Aut (Γ) is isomorphic to S_n. Assume that there is a link (f, α, β) in SYM (Γ) since (f, α, β) is a link, then α and β are distinct vertices of SYM (Γ), let I_u be a loop in Γ_E, then $\sigma f(I_u)=\alpha\ (\sigma(I_u))=\alpha(u)$ and $\tau f(I_u)=\beta\ (\tau(I_u))=\beta(u)$, but since $\beta \neq \alpha$, this implies that $\alpha(u) \neq \beta(u)$ for some $u \in \Gamma$, and since Γ is discrete, then F is not bijective on Γ, and this contradicts that there is a link in SYM (Γ) so it is very clear that all edges of SYM (Γ) are loops, in other words SYM (Γ) is discrete.

Before we prove the next theorem we first recall the definition of the wreath product $G \wr z_2$ of a group G by the cyclic group z_2 of order 2. It is the semi-direct product or sometimes it is called the split extention of $G \times G$ by z_2, where z_2 acts on $G \times G$ by conjugation ($G \times G$ is the direct product of G by G).

Theorem. Let Δ_2 denote a digraph consisting of two copies of a 1-complete digraph Γ on n vertices. Then Aut (Δ_2) is isomorphic to $S_n \wr z_2$.

Proof. Let Γ be represented by the following diagram then Δ_2 can be represented by the diagram



(Γ and Γ_1 are the same but they have different names to distinguish between them).

Any automorphism α of Γ is an automorphism of Δ_2, where α fixes the edges of Γ_1, similarly for the automorphisms of Γ_1, and since $\mathrm{Aut}(\Gamma)$ is isomorphic to S_n then we have $S_n \times S_n$ automorphisms of Δ_2. Now we define an automorphism h of Δ_2 such that $h^2 = i$ the identity automorphism. First let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be vertices of Γ and Γ_1 respectively, let e_{ij} and e'_{ij} be the edges of Γ and Γ_1 respectively, where e_{ij} is the edge of Γ which connects a_i to a_j, $i \neq j$, and e'_{ij} is the edge of Γ_1 which connects b_i, b_j, the automorphism h is defined as follows :

$$h(e_{ij}) = e'_{ij}, \quad h(e'_{ij}) = e_{ij}, \quad h(I_{a_i}) = I_{b_1}$$

and

$$h(I_{b_i}) = I_{a_l}, \; i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, n.$$

Let $\alpha \in \mathrm{Aut}(\Gamma)$ and $\beta \in \mathrm{Aut}(\Gamma_1)$, then it can easily be shown that, $h^{-1}\,\alpha\, h \in \mathrm{Aut}(\Gamma_1)$ and $h^{-1}\,\beta\, h \in \mathrm{Aut}(\Gamma)$, also the rest of the automorphisms of Δ_2 are $\{k\,h : k \in \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Gamma_1)\}$. So $\mathrm{Aut}(\Delta_2)$ is isomorphic to the split extention $(S_n \times S_n) \rtimes Z_n$ of $S_n \times S_n$ by Z_2 which is $S_n \wr Z_2$.

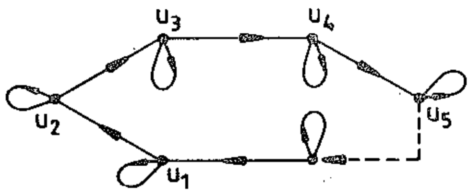
Let Γ be an n-complete circuit digraph (that is a digraph consisting of n vertices with a loop at each vertex and one directed edge e_{ij} between any two vertices u_i and u_j such that

$$\tau(e_{ij}) = \sigma(e_{i+1\, j+1}), \; i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, n$$

then we have the following theorem :

Theorem. Let Γ be an n-circuit digraph. Then $\mathrm{SYM}(\Gamma)$ is discrete (that is all edges are loops) on n vertices.

Proof. First the digraph Γ can be represented by the following diagram:



It is very clear that $\mathrm{Aut}(\Gamma)$ is isomorphic to the cyclic group Z_n of order n.

Assume that there is a link $(f, \alpha, \beta) \in \mathrm{SYM}(\Gamma)$ and let e_{ij} be the edge which connects u_i and u_j then we can find an edge x in Γ such that $\sigma \circ f(x) = u_j$ and $\tau \circ f(x) = u_j$, but since Γ is not complete then there is no edge from u_j to u_i and this contradicts the assumption. Hence all the edges of $\mathrm{SYM}(\Gamma)$ are loops of the form (α, α, α), $\alpha \in \mathrm{Aut}(\Gamma)$.

Proposition. If Γ is a discrete digraph, then $\mathrm{SYM}(\Gamma)$ is discrete and it is isomorphic to $\mathrm{Aut}(\Gamma)$.

Proof. Aut(Γ) is S_n, and we follow the same technique of the above theorem for the rest of the proof.

Now we come to the group graph concept which has been studied in detail by Loday [3] and Ribenboim [4]. A group graph is defined by Loday (using here the notation of [2]) to be a group G with two endomorphisms s, t of G such that $st = t$, $ts = s$.

For any digraph Γ we can construct a digraph SYM(Γ). The monoid multiplication on END(Γ) gives SYM(Γ) a group structure, and it can easily be checked that the two structures on SYM(Γ) are compatible, in the sense that the source and target functions are group homomorphisms.

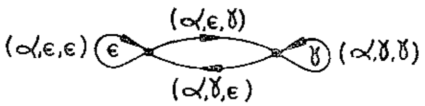
Theorem. Let N be a normal subgroup of a group G, then there are two group homomorphisms s, t of G such that $(G; s, t)$ is a group graph.

Proof. Let $\{N, g_1 N, g_2 N, \ldots, g_r N\}$ be the cosets of N in G. Define $s = t: G \longrightarrow G$ by

$s(g_i n_j) = g_i$, where $g_i \in G$ and $n_j \in N$, so $s(g_i n_j g_p n_q) = s(g_i g_p n_j' n_q) =$
$= g_i g_p = s(g_i n_j)\, s(g_p n_q)$ where $n_j', n_q \in N$ and $g_p \in G$.

We end this paper with the idea of the subgroup graph. Let (G, s, t) be a group graph, then $(H, s|_H, t|_H)$ is a subgroup graph of $(G; s, t)$, iff $s(H)$ and $t(H)$ are subgroups of H where H is a subgroup of G and $s|_H, t|_H$ are the restrictions of s, t on H.

Example 5. Consider example 4, then we have the following two subgroup graphs :



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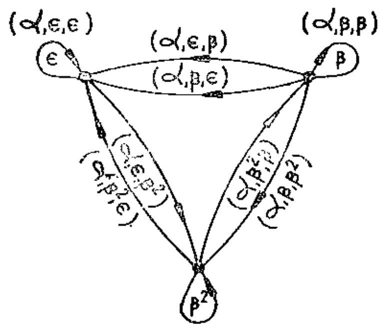

(2)

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ÖZET

Bu çalışmada, bazı Γ digrafları için SYM (Γ) nın bazı özelikleri incelenmekte, Aut (Γ) ve SYM (Γ) ile ilgili bazı teoremler verilmekte ve ayrıca, graf gruplarına ilişkin bir teorem ispat edilmektedir.