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A NOTE ON MORPHISM GRAPHS

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The isomorphic relationship

$$\mathrm{Mor}\,(X \times Y, Z) = \mathrm{Mor}\,(X, \mathrm{MG}\,(Y, Z))$$

between the set of morphisms from $X \times Y$ to Z and the set of morphisms from X to the morphism graph $\mathrm{MG}\,(Y, Z)$, where X, Y and Z are graphs, has been used in [1]. Here we discuss this relationship for certain directed graphs by using the posets with Hasse diagrams. Also some theorems related to morphism graphs have been proved.

DEFINITIONS

1. A directed graph (Diagraph) X consists of two disjoint sets X_V and X_E, called the set of vertices and the set of edges respectively, and two functions, $s, t : X_E \longrightarrow X_V$, called the source and target maps respectively. It is sometimes convenient to distinguish between those edges where the source and target maps coincide and those where they differ. A loop is an edge e such that $se = te$, and a link is an edge e such that $se \neq te$.

For the purpose of this paper we use an alternative, algebraic definition of a diagraph, namely, a set X with two functions $s, t : X \longrightarrow X$ such that $ts = s$ and $st = t$, it is easily shown that this definiton implies that $s^2 = s$, $t^2 = t$, and Image (s)=Image (t), thus we can take X=Image (s)= Image (t), $X_E = X - X_V$ (this definition has been used in [3]).

Example 1.

x	$s(x)$	$t(x)$	
u	u	u	$V(X)$
v	v	v	
w	w	w	
z	z	z	
a	u	u	
b, c	z	z	
d, e	u	v	
f	w	v	
g	v	w	
h	u	z	

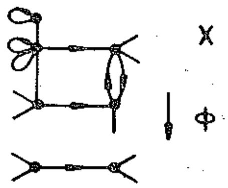
2. The set of morphisms, Mor (X, Y) between directed graphs X and Y is the set of functions

$$\phi : X \longrightarrow Y$$

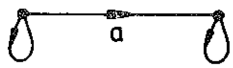
which satisfy $\phi\, s(x) = s(\phi(x))$ and $\phi(t(x)) = t(\phi(x))$.

Note. A morphism may be illustrated by a "3-dimensional" sketch in which the inverse images of vertices and edges lie directly above their images.

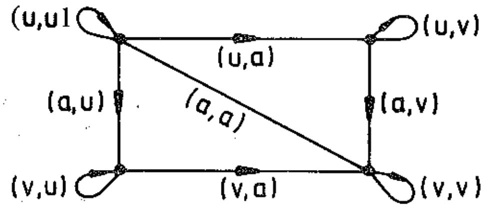
Example 2.



3. The product $X \times Y$ of two diagraphs X and Y is defined by
$X \times Y = \{(x, y) : x \in X, y \in Y, s(x, y) = (s(x), s(y)), t(x, y) = (t(x), t(y))\}$.

Example 3. Let $X = Y =$ 

, then

$X \times Y =$ 

4. Let $\phi, \psi \in$ Mor (X, Y), then $\mathrm{Con}_{\phi,\psi}(X, Y)$ (the connecting maps) is the set of maps $\alpha : X \longrightarrow Y$ satisfying

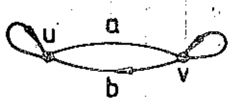
$$\left.\begin{aligned} s\,\alpha(x) &= \phi\, s(x) \\ t\,\alpha(x) &= \psi\, t(x) \end{aligned}\right\}, \text{ for all } x \text{ in } X.$$

Such an α is called a (ϕ, ψ)-connector.

Note. ϕ is a (ϕ, ϕ)-connector for all $\phi \in$ Mor (X, Y), since $s\,\phi(x) = s(x)$ and $t\phi(x) = t(x)$.

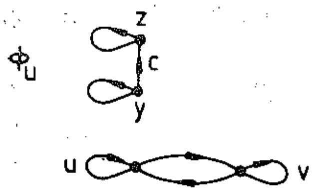
5. The morphism graph MG (X, Y) is the set of triples

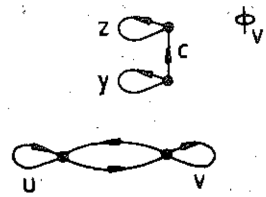
$$\{(\alpha, \phi, \psi) : \phi, \psi \in \mathrm{Mor}\,(X, Y), \alpha \in \mathrm{Con}_{\phi,\psi}(X, Y)\}.$$

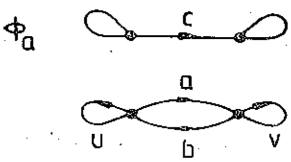
Example 4. Let $Y =$ 

, $X =$ 

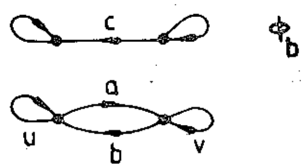
then, Mor $(X, Y) = \{\phi_u, \phi_v, \phi_a, \phi_b\}$ is represented by the following diagrams :



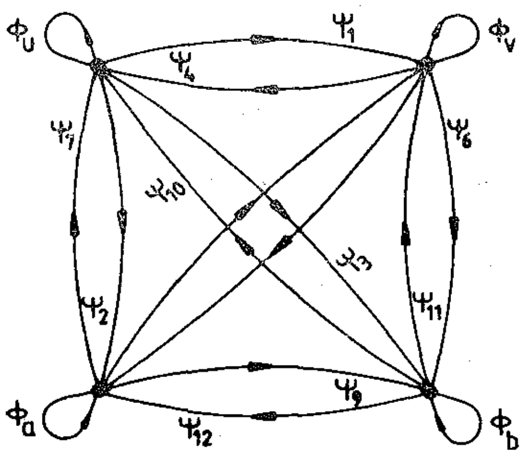




ϕ	$\phi(y)$	$\phi(c)$	$\phi(z)$
ϕ_u	u	u	u
ϕ_v	v	v	v
ϕ_a	u	a	v
ϕ_b	v	b	u



x	$\phi_u s(x)$	$\phi_u t(x)$	$\phi_v s(x)$	$\phi_v t(x)$	$\phi_a s(x)$	$\phi_a t(x)$	$\phi_b s(x)$	$\phi_b t(x)$
y	u	u	v	v	u	u	v	v
c	u	u	v	v	u	v	v	u
z	u	u	v	v	v	v	u	u



ψ	$\psi(y)$	$\psi(c)$	$\psi(z)$
ψ_1	a	a	a
ψ_2	u	a	a
ψ_3	a	u	u
ψ_4	b	b	b
ψ_5	b	v	v
ψ_6	v	b	b
ψ_7	u	u	b
ψ_8	a	a	v
ψ_9	a	u	b
ψ_{10}	b	b	u
ψ_{11}	v	v	a
ψ_{12}	b	v	a

6. If Γ is any graph, its Hasse diagram is the graph $\Gamma^* = \Gamma_V \cup E'$ where E' is the set of all pairs (x, y) with $x, y \in \Gamma_V$, $x \neq y$ and $\text{Max}\{L(\eta) \mid \eta \in P(\Gamma), s(\eta) = x, t(\eta) = y\} = 1$, where $L(\eta), \eta$ and $P(\Gamma)$ are defined as follows:

$L(\eta)$ is the length of a path η where a path of length $n \geqslant 1$ is an n-tupie $(y_n, \ldots, y_1)$, $y_i \in \Gamma_E$.

We denote by $P(\Gamma)$ the set of irreducible paths of Γ, where an irreducible path of Γ is a path of length 0 or 1, or any path η, $\eta = (y_n, \ldots, y_1)$, with $n \geqslant 2$, such that the vertices $s(y_1), s(y_2), \ldots, s(y_n)$ are all distinct. For more details of Hasse diagrams, see [2].

Theorem. Let X and Y be two diagraphs, if Y is 1-complete (i.e. it has exactly 1-directed edge from any vertex to any other) then MG(X, Y) contains a copy of Y.

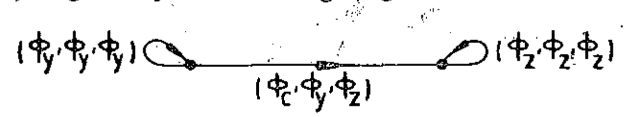
Proof. Let $v_1, v_2, \ldots, v_r$ be the vertices of Y, then $\{\phi_{v_i}\}_{i=1}^{r} \subseteq \text{Mor}(X, Y)$, where each ϕ_{v_i} is defined by $\phi_{v_i}(x) = x$ for all edges $x \in X$, define the constant connectors $\{\psi_y, y \in Y\}$ where $\psi_y(x) = y$ for all x in X, and for all edges y in Y. These connectors give the set of triples $\{(\psi_y, \phi_{v_i}, \phi_{v_j})\}$ where $v_i = s(y)$ and $v_j = t(y)$. For each vertex $v_i \in Y$, $\psi_{v_i} = \phi_{v_i}$, and this completes the proof. Now we discuss MG(X, Y) in example 3. First, $\text{Mor}(X, Y) = \{\phi_y, \phi_z\}$ where

ϕ	u	v	a	b
ϕ_y	y	y	y	y
ϕ_z	z	z	z	z

also we have the following table :

x	$\phi s(x)$	$\phi t(y)$	$\phi s(x)$	$\phi t(x)$
u	y	y	z	z
v	y	y	z	z
a	y	y	z	z
b	y	y	z	z

So, MG(Y, X) is given by the following diagram :



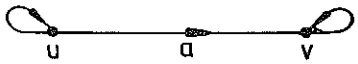
There is no (ϕ_z, ϕ_y)-connector because we require such an α to satisfy $s(\alpha(x)) = z$ and $t(\alpha(x)) = y$ for all $x \in Y$, and there is no edge from z to y in X.

Proposition. MG(X, Y) is not isomorphic to MG(Y, X).

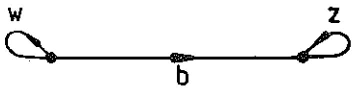
Proof. See the above example.

An illustration of Mor$((X \times Y), Z) \cong$ Mor$(X, \mathrm{MG}(Y, Z))$

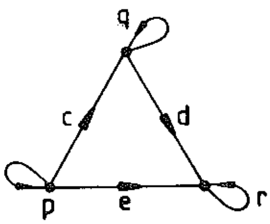
Take $X =$



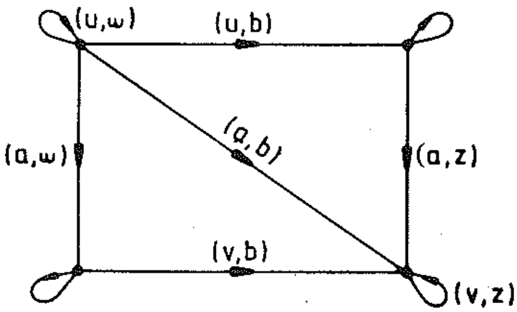
$Y =$



and $Z =$



then $X \times Y =$

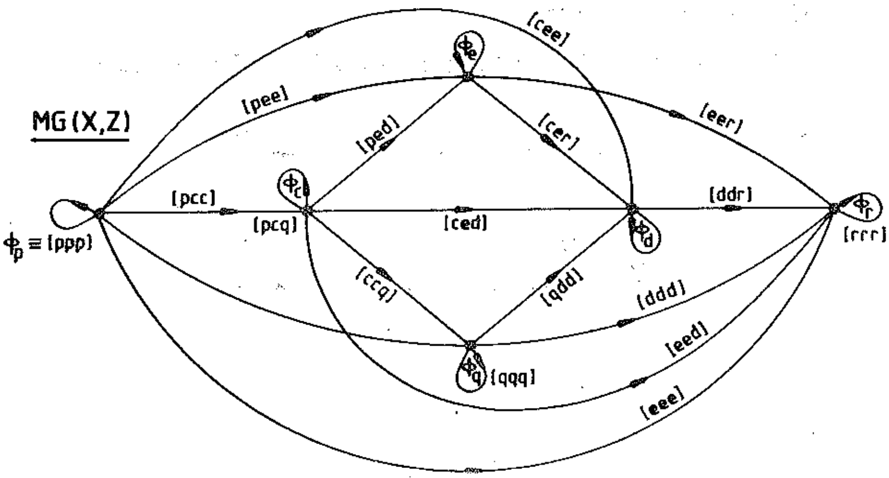


and Mor$(X, Z) = \{\phi_p, \phi_q, \phi_r, \phi_c, \phi_d, \phi_e\}$ where ϕ_x maps a to x.

To construct MG(X, Z) we need the following table:

x	$\phi(u)$	$\phi(a)$	$\phi(v)$	sources			targets		
p	p	p	p	p	p	p	p	p	p
q	q	q	q	q	q	q	q	q	q
r	r	r	r	r	r	r	r	r	r
c	p	c	q	p	p	q	p	q	q
d	q	d	r	q	q	r	q	r	r
e	p	e	r	p	p	r	p	r	r

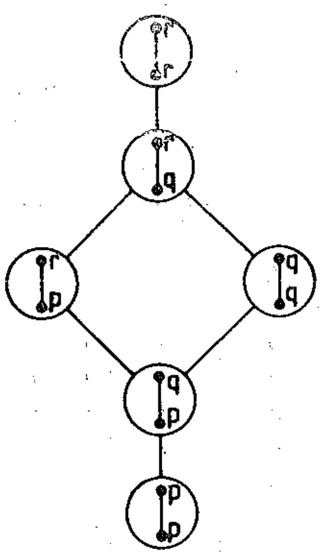
The diagram of MG(X, Z) is presented below.



Note. Z can be considered as a directed graph associated to the poset (Hasse diagram):

r
q
p

Similarly, X is associated to Hasse diagram v / u, then MG(X, Z) is associated to Hasse diagram:

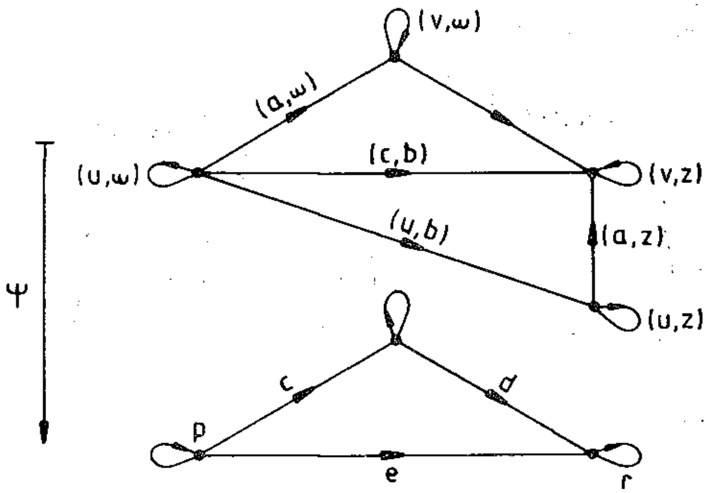


It is clear that the morphism diagraphs can be viewed as a morphism of posets.

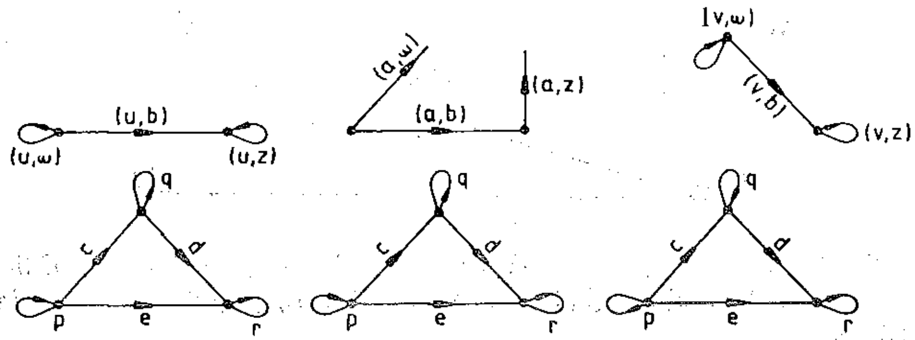
Now we construct Mor$(X, \mathrm{MG}(Y, Z))$. The morphism graphs MG(Y, Z) and MG(X, Z) are isomorphic, for, replace u, a, v everywhere by w, b, z.

Here we need a diagraph morphism from X to MG(Y, Z), since MG(Y, Z) has twenty edges, there are twenty such morphisms drawn on page q (List A).

The second part of the isomorphic relationship is Mor$(X \times Y, Z)$ and this must contain twenty morphisms, for Mor$(X, \mathrm{MG}(Y, Z))$ does. Consider one particular morphism ψ.



If we split the edges of $X \times Y$ into sets $\{(u, b), (u, w), (u, z)\}$, $\{(a, w), (a, b), (a, z)\}$ and $\{(v, w), (v, b), (v, z)\}$, then we have the following diagrams:



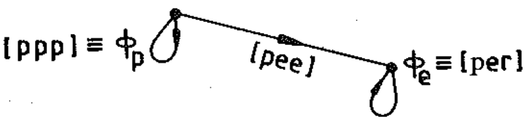
From these diagrams we have

$$\left.\begin{matrix} w \longrightarrow p \\ b \longrightarrow e \\ z \longrightarrow r \end{matrix}\right\} [per] \equiv \phi_e, \quad \left.\begin{matrix} w \longrightarrow c \\ b \longrightarrow e \\ z \longrightarrow r \end{matrix}\right\} [cer], \quad \left.\begin{matrix} w \longrightarrow q \\ b \longrightarrow d \\ z \longrightarrow r \end{matrix}\right\} [qdr] \equiv \phi_d$$

Thus ψ gives the diagram 

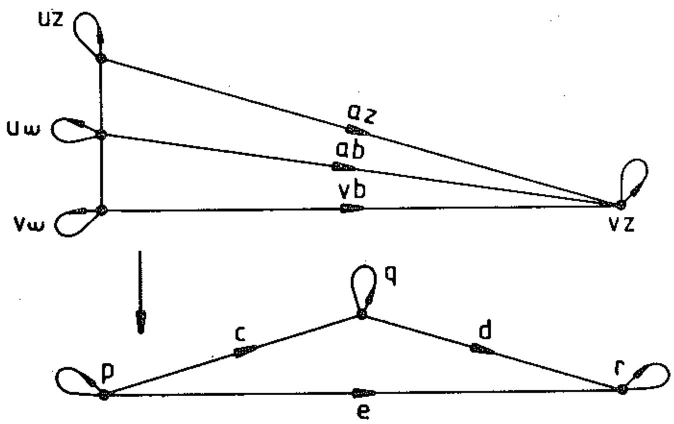
and this corresponds to a morphism in Mor$(X, \mathrm{MG}(Y, Z))$ which is number 13 in (List A).

To illustrate the reverse process, choose a morphism from X to $\mathrm{MG}(Y, Z)$, number 3 in (List A) say,



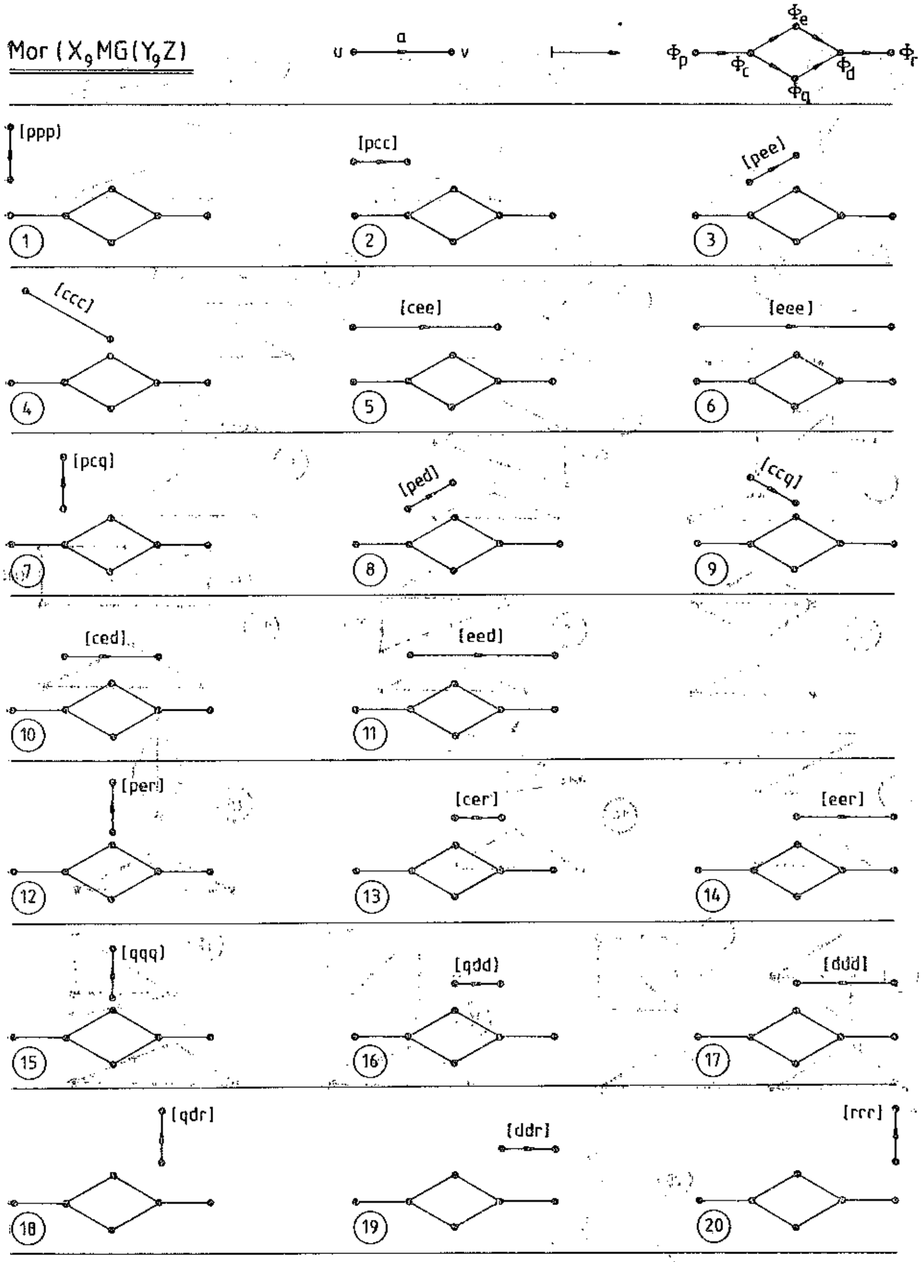
$$\text{so } u \longrightarrow \begin{pmatrix} w \longrightarrow p \\ b \longrightarrow p \\ z \longrightarrow p \end{pmatrix}, \; a \longrightarrow \begin{pmatrix} w \longrightarrow p \\ p \longrightarrow e \\ z \longrightarrow e \end{pmatrix}, \; v \longrightarrow \begin{pmatrix} w \longrightarrow p \\ b \longrightarrow e \\ z \longrightarrow r \end{pmatrix}.$$

This enables us to sketch the following projection diagram (where we shorten (u, w) to uw, etc).

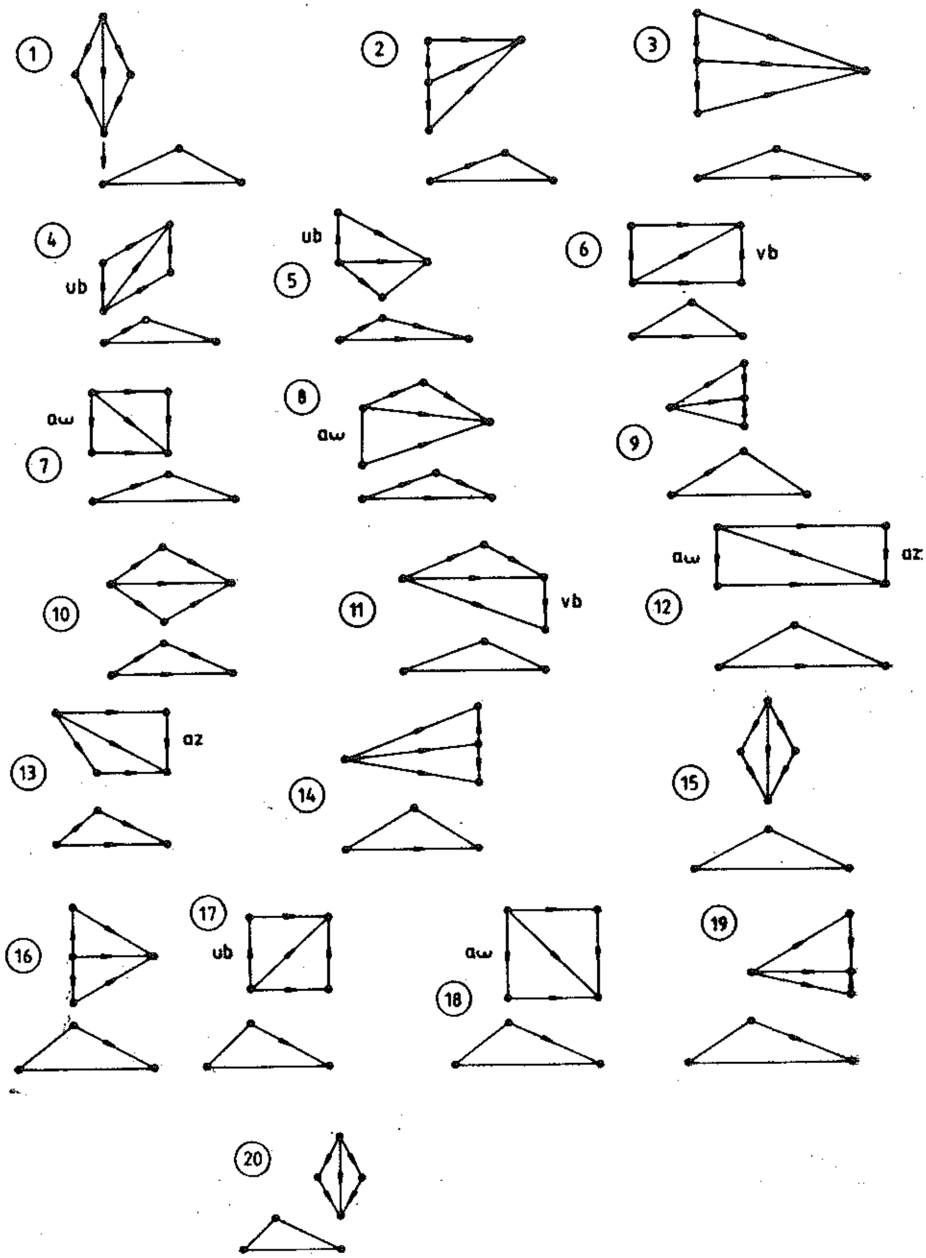


On the following page we sketch a projection diagram for each of the morphisms of Mor$(X, \mathrm{MG}(Y, Z))$, so we have twenty projection diagrams in (List B).

(List A)


Mor (X, MG(Y, Z))
u
a
v
Φp
Φc
Φe
Φq
Φd
Φr
[ppp]
[pcc]
[pee]
1
2
3
[ccc]
[cee]
[eee]
4
5
6
[pcq]
[ped]
[ccq]
7
8
9
[ced]
[eed]
10
11
[per]
[cer]
[eer]
12
13
14
[qqq]
[qdd]
[ddd]
15
16
17
[qdr]
[ddr]
[rrr]
18
19
20

(List B)


1
2
3
4
ub
ub
5
6
vb
aw
7
8
aw
9
10
11
vb
12
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az
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az
14
15
16
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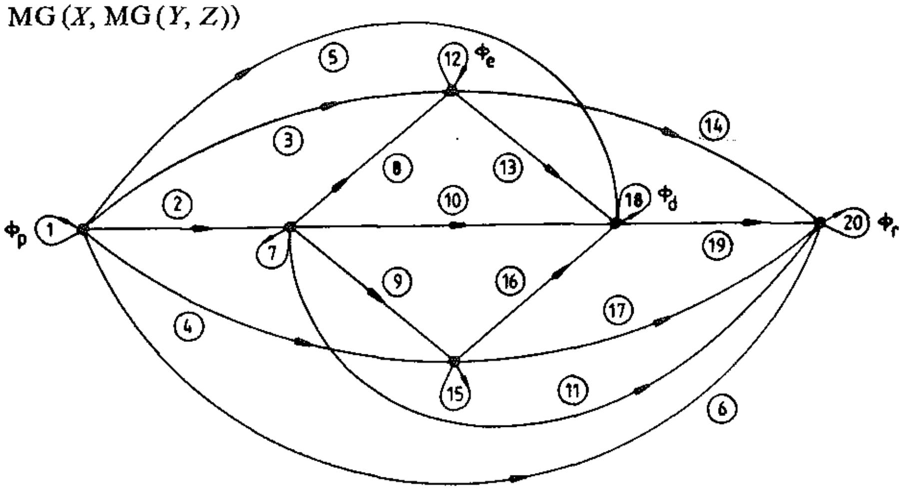
18
19
20

MG$(X, \mathrm{MG}(Y, Z))$



i	$\phi_i s(y)$	$\phi_i s(c)$	$\phi_i s(z)$
1	ϕ_p	ϕ_p	ϕ_p
2	ϕ_p	ϕ_p	ϕ_c
3	ϕ_p	ϕ_p	ϕ_e
4	ϕ_p	ϕ_p	ϕ_q
5	ϕ_p	ϕ_p	ϕ_d
6	ϕ_p	ϕ_p	ϕ_r
7	ϕ_c	ϕ_c	ϕ_c
8	ϕ_c	ϕ_c	ϕ_e
9	ϕ_c	ϕ_c	ϕ_q
10	ϕ_c	ϕ_c	ϕ_d
11	ϕ_c	ϕ_c	ϕ_r
12	ϕ_e	ϕ_e	ϕ_e
13	ϕ_e	ϕ_e	ϕ_d
14	ϕ_e	ϕ_e	ϕ_r
15	ϕ_q	ϕ_q	ϕ_q
16	ϕ_q	ϕ_q	ϕ_d
17	ϕ_q	ϕ_q	ϕ_r
18	ϕ_d	ϕ_d	ϕ_d
19	ϕ_d	ϕ_d	ϕ_r
20	ϕ_r	ϕ_r	ϕ_r

i	$\phi_i t(y)$	$\phi_i t(c)$	$\phi_i t(z)$
1	ϕ_p	ϕ_p	ϕ_p
2	ϕ_p	ϕ_c	ϕ_c
3	ϕ_p	ϕ_e	ϕ_e
4	ϕ_p	ϕ_q	ϕ_q
5	ϕ_p	ϕ_d	ϕ_d
6	ϕ_p	ϕ_r	ϕ_r
7	ϕ_c	ϕ_c	ϕ_c
8	ϕ_c	ϕ_e	ϕ_e
9	ϕ_c	ϕ_q	ϕ_q
10	ϕ_c	ϕ_d	ϕ_d
11	ϕ_c	ϕ_r	ϕ_r
12	ϕ_e	ϕ_e	ϕ_e
13	ϕ_e	ϕ_d	ϕ_d
14	ϕ_e	ϕ_r	ϕ_r
15	ϕ_q	ϕ_q	ϕ_q
16	ϕ_q	ϕ_d	ϕ_d
17	ϕ_q	ϕ_r	ϕ_r
18	ϕ_d	ϕ_d	ϕ_d
19	ϕ_d	ϕ_r	ϕ_r
20	ϕ_r	ϕ_r	ϕ_r