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QUADRATIC MEAN FUNCTION OF ENTIRE DIRICHLET SERIES

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Let E be the set of all entire functions $f(s) = \sum_{n \in N} a_n e^{s\lambda_n}$ defined by an everywhere convergent Dirichlet series whose exponents are subjected to the condition $\limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = D \in R_+ \cup \{0\}$ (R_+ is the set of positive reals). Also let $I_2(\sigma, f)$ be the quadratic mean function of an $f \in E$, on $\mathrm{Re}(s) = \sigma$, defined as $I_2(\sigma, f) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 \, dt$. In this paper we have studied a few results pertaining to the function I_2.

1. Let E be the set of mappings $f: C \to C$ (C is the complex field) such that the image under f of an element $s \in C$ is $f(s) = \sum_{n \in N} a_n e^{s\lambda_n}$ with $\limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = D \in R_+ \cup \{0\}$ (R_+ is the set of positive reals), and $\sigma_c^f = +\infty$ (σ_c^f is the abscissa of convergence of the Dirichlet series defining f); N is the set of natural numbers 0, 1, 2,..., $< a_n \mid n \in N >$ is a sequence in C, $s = \sigma + it$, σ, $t \in R$ (R is the field of reals), and $< \lambda_n \mid n \in N >$ is a strictly increasing unbounded sequence of nonnegative reals. Since the Dirichlet series defining f converges for each $s \in C$, f is an entire function. Also, since $D \in R_+ \cup \{0\}$, we have ([1], p. 168), $\sigma_a^f = +\infty$ (σ_a^f is the abscissa of absolute convergence of the Dirichlet series defining f) and that f is bounded on each vertical line $\mathrm{Re}(s) = \sigma_0$.

Let $f \in E$ be an entire function. The maximum modulus function M of f on any vertical line $\mathrm{Re}(s) = \sigma$, is defined as

$$M(\sigma, f) = \sup_{t \in R} \{ |f(\sigma + it)| \}, \quad \forall \sigma < \sigma_c^f, \tag{1.1}$$

the maximum term function μ, for $\mathrm{Re}(s) = \sigma$, in the Dirichlet series defining f, is defined as

$$\mu(\sigma, f) = \max_{n \in N} \{ |a_n| e^{\sigma\lambda_n} \}, \quad \forall \sigma < \sigma_c^f, \tag{1.2}$$

and the quadratic mean function I_2 of f, on $\mathrm{Re}(s)=\sigma$, is defined as

$$I_2(\sigma, f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2\, dt, \quad \forall\, \sigma < \sigma_c^f. \tag{1.3}$$

In this paper we study a few results regarding the function I_2.

2. First we show that :

Theorem 1. If $f, g \in E$ are two entire functions such that for any $s \in C$, $f(s) = \sum_{m\in N} a_m e^{s\lambda_m}$, $g(s) = \sum_{n\in N} b_n e^{s\mu_n}$, and h is the Dirichlet product of f and g, i.e. for any $s\in C$, $h(s) = \sum_{p\in N} c_p e^{s\nu_p}$ where $c_p = \sum_{\lambda_m+\mu_n=\nu_p} a_m b_n$, then $h\in E$, and

$$\mu(\sigma, h) < (I_2(\sigma, f)\ I_2(o, g))^{1/2}. \tag{2.1}$$

Proof. $h\in E$ follows from the fact [2] that E is an algebra. Also, we have

$$\begin{aligned} |c_p|\, e^{\sigma\nu_p} &= \Big| \sum_{\lambda_m+\mu_n=\nu_p} a_m b_n \Big|\, e^{\sigma\nu_p} \\ &\le \sum_{\lambda_m+\mu_n=\nu_p} |a_m|\, |b_n|\, e^{\sigma(\lambda_m+\mu_n)} \\ &\le \Big(\sum_{m\le p} |a_m|^2 e^{2\sigma\lambda_m}\Big)^{1/2} \Big(\sum_{n\le p} |b_n|^2 e^{2\sigma\mu_n}\Big)^{1/2} \\ &< (I_2(\sigma, f)\ I_2(\sigma, g))^{1/2}, \end{aligned}$$

in view of the fact ([3], formula (2.2)) that for any $f\in E$, $I_2(\sigma, f) = \sum_{n\in N} |a_n|^2 e^{2\sigma\lambda_n}$. Since the last inequality is true for all p, it follows that

$$\mu(\sigma, h) < (I_2(\sigma, f)\ I_2(\sigma, g))^{1/2}.$$

We give below two interesting applications of (2.1).

i) If f, g, h are of Ritt orders ρ_1, ρ_2, and ρ, respectively, then

$$\rho \le \rho_1 + \rho_2\,; \tag{2.2}$$

a result established otherwise by the first author ([4], Theo. 1).

The result in (2.2) follows from (2.1) and the following facts : a) that for any entire function $f\in E$ of Ritt order $\rho \in R_+^* \cup \{0\}$, in view of ([5], Theo. 5), ([6], Theos. 2.7 and 2.8), and ([3], Theo. 3), respectively,

$$\rho = \limsup_{\sigma \to +\infty} \frac{\log \log M(\sigma, f)}{\sigma} = \limsup_{\sigma \to +\infty} \frac{\log \log \mu(\sigma, f)}{\sigma} = \limsup_{\sigma \to +\infty} \frac{\log \log I_2(\sigma, f)}{\sigma}, \tag{2.3}$$

and b) that ([3], Theo. 1) $\log I_2(\sigma, f)$ is an increasing convex function of σ.

Remark. The result in the last equality in (2.3) although has been established for entire function $f \in E$ of Ritt order $\rho \in R_+$ and for $D = 0$, but by a slight modification in the argument the result holds for any $f \in E$.

ii) If f, g, h are of the same Ritt order $\rho \in R_+$ and types T_1, T_2 and T, respectively, then

$$T \leq T_1 + T_2 ; \tag{2.4}$$

a result established otherwise by the first author ([4], Theo. 2).

As previously, the result in (2.4) follows from (2.1), in view of the fact, that for any entire function $f \in E$ of Ritt order $\rho \in R_+$ and type $T \in R_+^* \cup \{0\}$, we have, in view of ([5], Theo. 5), ([7], Theo. 5), and ([3], Theo. 3), respectively,

$$T = \limsup_{\sigma \to +\infty} \frac{\log M(\sigma, f)}{e^{\rho\sigma}} = \limsup_{\sigma \to +\infty} \frac{\log \mu(\sigma, f)}{e^{\rho\sigma}} = \limsup_{\sigma \to +\infty} \frac{\frac{1}{2} \log I_2(\sigma, f)}{e^{\rho\sigma}}. \tag{2.5}$$

Remark. The result in the last equality has been proved under the condition that $D = 0$, but is true for any $D \in R_+ \cup \{0\}$.

Next we find that:

Theorem 2. If $f \in E$ is an entire function of Ritt order $\rho \in R_+$ and perfectly regular growth and type $T \in R_+$, then

$$\lim_{\sigma \to +\infty} \frac{I_2'(\sigma, f)}{I_2(\sigma, f)\, \rho\, T e^{\rho\sigma}} = 2, \tag{2.6}$$

where $I_2'(\sigma, f)$ denotes the derivative of $I_2(\sigma, f)$ with respect to σ.

Proof. From (2.5), we get for any $\varepsilon \in R_+$ and sufficiently large σ,

$$(2T - \varepsilon)\, e^{\rho\sigma} < \log I_2(\sigma, f) < (2T + \varepsilon)\, e^{\rho\sigma}. \tag{2.7}$$

Also, since $\log I_2(\sigma, f)$ is an increasing convex function of σ, we may write for any $\sigma, \sigma_0\ (\sigma > \sigma_0)$,

$$\log I_2(\sigma, f) = \log I_2(\sigma_0, f) + \int_{\sigma_0}^{\sigma} \frac{I_2'(x, f)}{I_2(x, f)}\, dx. \tag{2.8}$$

Now, for any $k \in R_+ \cup \{0\}$, we have

$$\int_{\sigma}^{\sigma+k} \frac{I_2'(x, f)}{I_2(x, f)}\, dx = \int_{0}^{\sigma+k} \frac{I_2'(x, f)}{I_2(x, f)}\, dx - \int_{0}^{\sigma} \frac{I_2'(x, f)}{I_2(x, f)}\, dx,$$

$$= \log I_2(\sigma + k, f) - \log I_2(\sigma, f), \text{ in view of (2.8)},$$

$$\leq (2T + \varepsilon)\, e^{\rho(\sigma+k)} - (2T - \varepsilon)\, e^{\rho\sigma}, \text{ in view of (2.7)},$$

$$= e^{\rho\sigma}\{2T(e^{\rho k} - 1) + \varepsilon(e^{\rho k} + 1)\}. \tag{2.9}$$

But

$$\int_{\sigma}^{\sigma+k} \frac{I_2'(x, f)}{I_2(x, f)}\, dx \geq \frac{I_2'(x, f)}{I_2(x, f)}\, k. \tag{2.10}$$

Hence, from (2.9) and (2.10),

$$\frac{I_2'(\sigma, f)}{I_2(\sigma, f)\, e^{\rho\sigma}} < \frac{2T(e^{\rho k} - 1) + \varepsilon(e^{\rho k} + 1)}{k}. \tag{2.11}$$

Since k is arbitrary but belongs to $R_+ \cup \{0\}$ and the left hand side of (2.11) is independent of k, it follows that

$$\limsup_{\sigma \to +\infty} \frac{I_2'(\sigma, f)}{I_2(\sigma, f)\, e^{\rho\sigma}} \leq 2\rho T. \tag{2.12}$$

Similarly, we can show that

$$\liminf_{\sigma \to +\infty} \frac{I_2'(\sigma, f)}{I_2(\sigma, f)\, e^{\rho\sigma}} \geq 2\rho T. \tag{2.13}$$

Hence the theorem.

Theorem 2 leads easily to the following well known fact :

Corollary 1. If $f \in E$ is an entire function of Ritt order $\rho \in R_+$ and is of perfectly regular growth and type $T \in R_+$, then it is of regular growth.

From (2.6) we find that

$$\text{iog}\left(\frac{I_2'(\sigma, f)}{I_2(\sigma, f)}\right) \sim \log 2\rho T + \rho\sigma.$$

Hence

$$\lim_{\sigma \to +\infty} \frac{\log (I_2'(\sigma, f) / I_2(\sigma, f))}{\sigma} = \rho,$$

showing that f is of regular growth, since ([8], Formula (7.3.13)),

$$\lim_{\sigma \to +\infty} \frac{\log (I_2'(\sigma, f) / I_2(\sigma, f))}{\sigma} = \lim_{\sigma \to +\infty} \frac{\log \log I_2(\sigma, f)}{\sigma}.$$

Remark. Since the lower type of entire functions in E of irregular growth is always zero ([9], p. 250), with the same argument, it can be shown that Theorem 2 holds for such functions also.

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ÖZET

E, üsleri $\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D \in R_+ \cup \{0\}$ koşulunu gerçekleyen, her yerde yakınsak bir Dirichlet serisi ile tanımlanan bütün $f(s) = \sum_{n\in N} a_n e^{s\lambda_n}$ tam fonksiyonlarının cümlesini göstersin. $I_2(\sigma, f)$ de $\mathrm{Re}(s) = \sigma$ üzerinde bir $f \in E$ nin $I_2(\sigma, f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2 \, dt$ şeklinde tanımlanan kuadratik ortalama fonksiyonu olsun. Bu çalışmada I_2 fonksiyonuna ilişkin bazı sonuçlar elde edilmektedir.