İstanbul Üniv. Fen Fak. Mat. Der. **51** (1992), 35-39

ON A TYPE OF P-SASAKIAN MANIFOLD*)

U.C. DE - N. GUHA

The object of this paper is to study a type of P-Sasakian manifold in which $R(X, Y) . C = 0$ where C is the conformal curvature tensor, R is the Riemannian curvature and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y.

INTRODUCTION

Let (M^n, g) be an n-dimensional Riemannian manifold admitting a 1-form η which satisfies the conditions

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0 \tag{1}$$

$$(\nabla_X \nabla_Y \eta)(Z) = -g(X, Z)\eta(Y) - g(X, Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z) \tag{2}$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. If moreover (M^n, g) admits a vector field ξ and a (1-1) tensor field ϕ such that

$$g(X, \xi) = \eta(X) \tag{3}$$

$$\eta(\xi) = 1 \tag{4}$$

$$\nabla_X \xi = \phi X \tag{5}$$

then such a manifold is called a para Sasakian manifold or briefly a P-Sasakian manifold by I. Sato and K. Matsumoto [1], [2]. This paper deals with a type of P-Sasakian manifold in which

$$R(X, Y) \,.\, C = 0 \tag{6}$$

where C is the conformal curvature tensor, R is the Riemannian curvature and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. In this connection we mention the work of T. Adati and T. Miyazawa [3], [4], T. Adati and K. Matsumoto [5] who studied P-Sasakian manifolds. Also we can mention the works of K. Sekigawa [6], Z. I. Szabo [7], L. Verstraelen [8], M. Petrovic-Torgasev and L. Verstraelen [9] who studied Riemannian manifolds and hyper surfaces of such manifolds satisfying the condition $R(X, Y) . R = 0$ or a condition similar to it.

*) 1980 Mathematics subject classification (AMS) : Primary 53C25

Let (M^n, g) be an n-dimensional Riemannian manifold admitting a 1-form η which satisfies the condition

$$(\nabla_X \eta)(Y) = -g(X, Y) + \eta(X)\eta(Y).$$

If moreover (M^n, g) admits a vector field ξ and a (1-1) tensor field ϕ such that conditions (3), (4) and (5) are satisfied, then it has been called a special P-Sasakian manifold or briefly a SP-Sasakian manifold [1], [2]. In this paper it is proved that if in a P-Sasakian manifold (M^n, g) $(n > 3)$ the relation (6) holds then the manifold is conformally flat and hence is an SP-Sasakian manifold. Also it is shown that a conformally symmetric P-Sasakian manifold (M^n, g) $(n > 3)$ is an SP-Sasakian manifold [n has been taken > 3 because it is known that $C = 0$, when $n = 3$].

1. PRELIMINARIES

It is known [1], [2] that in a P-Sasakian manifold the following relations hold:

$$\phi\,\xi = 0 \tag{1.1}$$

$$\phi^2 X = X - \eta(X)\,\xi \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y) \tag{1.3}$$

$$S(X, \xi) = -(n-1)\,\eta(X) \tag{1.4}$$

$$\eta(R(X, Y)Z) = g(X, Z)\,\eta(Y) - g(Y, Z)\,\eta(X) \tag{1.5}$$

$$R(\xi, X)\,Y = \eta(Y)\,X - g(X, Y)\,\xi \tag{1.6}$$

$$R(\xi, X)\,\xi = X - \eta(X)\,\xi \tag{1.7}$$

$$R(X, Y)\,\xi = \eta(X)\,Y - \eta(Y)\,X. \tag{1.8}$$

The above results will be used in the next section.

2. P-SASAKIAN MANIFOLD SATISFYING $R(X, Y) . C = 0$

We have

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}\,[g(Y, Z)\,QX - g(X, Z)\,QY + S(Y, Z)\,X -$$

$$-S(X, Z)\,Y] + \frac{r}{(n-1)(n-2)}\,[g(Y, Z)X - g(X, Z)Y] \tag{2.1}$$

where S is the Ricci tensor, r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S [10], i.e.

$$g(QX, Y) = S(X, Y).$$

Therefore

$$\eta(C(X, Y)Z) = g(C(X, Y)Z, \xi) =$$
$$= \frac{1}{n-2}\left[\left(\frac{r}{n-1}+1\right)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)) - \right.$$
$$\left. - (S(Y, Z)\eta(X) - S(X, Z)\eta(Y))\right]. \tag{2.2}$$

Putting $Z = \xi$ in (2.2) we get

$$\eta(C(X, Y)\xi) = 0. \tag{2.3}$$

Again putting $X = \xi$ in (2.2) we get

$$\eta(C(\xi, Y)Z) = \frac{1}{n-2}\left[\left(1+\frac{r}{n-1}\right)g(Y, Z) - S(Y, Z) - \right.$$
$$\left. - \left(\frac{r}{n-1}+n\right)\eta(Y)\eta(Z)\right]. \tag{2.4}$$

Now

$$(R(X, Y).C)(U, V)W = R(X, Y)C(U, V)W - C(R(X, Y)U, V)W -$$
$$- C(U, R(X, Y)V)W - C(U, V)R(X, Y)W.$$

In virtue of (6) we get

$$R(X, Y)C(U, V)W - C(R(X, Y)U, V)W -$$
$$- C(U, R(X, Y)V)W - C(U, V)R(X, Y)W = 0. \tag{2.5}$$

Therefore

$$g(R(\xi, Y)C(U, V)W, \xi) - g(C(R(\xi, Y)U, V)W, \xi) -$$
$$- g(C(U, R(\xi, Y)V)W, \xi) - g(C(U, V)R(\xi, Y)W, \xi) = 0. \tag{2.6}$$

From this it follows that

$$'C(U, V, W, Y) - \eta(Y)\eta(C(U, V)W) + \eta(U)\eta(C(Y, V)W) +$$
$$+ \eta(V)\eta(C(U, Y)W) + \eta(W)\eta(C(U, V)Y) -$$
$$- g(Y, U)\eta(C(\xi, V)W) - g(Y, V)\eta(C(U, \xi)W) -$$
$$- g(Y, W)\eta(C(U, V)\xi) = 0 \tag{2.7}$$

where

$$'C(U, V, W, Y) = g(C(U, V)W, Y).$$

Putting $Y = U$ in (2.7) we get

$$C(U,V,W,U)-\eta(U)\eta(C(U,V)W)+\eta(U)\eta(C(U,V)W)+ \\ +\eta(V)\eta(C(U,U)W)+\eta(W)\eta(C(U,V)U)- \\ -g(U,U)\eta(C(\xi,V)W)-g(U,V)\eta(C(U,\xi)W)- \\ -g(U,W)\eta(C(U,V)\xi)=0. \quad (2.8)$$

Let $\{e_i\}$, $i=1,2,\dots,n$ be an orthonormal basis of the tangent space at any point. Then the sum $1\leq i\leq n$ of the relation (2.8) for $U=e_i$ gives

$$\eta(C(\xi,V)W)=0. \quad (2.9)$$

Using (2.3) we have from (2.7)

$$C(U,V,W,Y)-\eta(Y)\eta(C(U,V)W)+\eta(U)\eta(C(Y,V)W)+ \\ +\eta(V)\eta(C(U,Y)W)+\eta(W)\eta(C(U,V)Y)- \\ -g(Y,U)\eta(C(\xi,V)W)-g(Y,V)\eta(C(U,\xi)W)=0. \quad (2.10)$$

In virtue of (2.4) and (2.9) we have

$$S(V,W)=\left(\frac{r}{n-1}+1\right)g(V,W)-\left(\frac{r}{n-1}+n\right)\eta(V)\,\eta(W). \quad (2.11)$$

Using (2.2), (2.3) and (2.11) the relation (2.10) reduces to

$$'C(U,V,W,Y)=0. \quad (2.12)$$

From (2.12) it follows that

$$C(U,V)W=0. \quad (2.13)$$

Thus we can state the following theorem :

Theorem 1. A P-Sasakian manifold (M^n,g) $(n>3)$ satisfying the relation $R(X,Y).C=0$ is conformally flat and hence is an SP-Sasakian manifold [11].

For a conformally symmetric Riemannian manifold [12] we have $\nabla C=0$. Hence for such a manifold $R(X,Y).C=0$ holds. Thus we have the following corollary of the above theorem :

Corollary. A conformally symmetric P-Sasakian manifold (M^n,g) $(n>3)$ is an SP-Sasakian manifold [13].

REFERENCES

[1] SATO, I. : *On a structure similar to the almost contact structure,* Tensor, N.S., **30** (1976), 219-224.

[2] SATO, I. and MATSUMOTO, K. : *On P-Sasakian manifolds satisfying certain conditions,* Tensor, N.S., 33 (1979), 173-178.

[3] ADATI, T. and MIYAZAWA, T. : *Some properties of P-Sasakian manifolds*, TRU Mathematics, **13-1** (1977), 33-42.

[4] ADATI, T. and MIYAZAWA, T. : *On P-Sasakian manifolds admitting some parallel and recurrent tensor*, Tensor, N.S., 33 (1979), 287-292.

[5] ADATI, T. and MATSUMOTO, K. : *On conformally recurrent and conformally symmetric P-Sasakian manifolds*, TRU Mathematics, **13-1** (1977), 25-32.

[6] SEKIGAWA, K. : *Almost Hermitian manifolds satisfying some curvature conditions*, Kodai Math. J., 2 (1979), 384-405.

[7] SZABO, Z.L : *Structure theorems on Riemannian spaces satisfying* $R(X, Y) . R = 0$, *I the local version*, J. Diff. Geom., 17 (1982), 531-582.

[8] VERSTRAELEN, L. : *A survey of relations between intrinsic and extrinsic properties for hypersurfaces*, Topics in Diff. Geom., Vol. I, II, Debrecen, 1984, 1279-1296.

[9] PETROVIC-TORGAREV, M. and VERSTRAELEN, L. : *On the concircular curvature tensor, the projective tensor and the Einstein curvature tensor of Bochner-Kahler manifolds*, Math. Rep. Toyama Univ., **10** (1987), 36-61.

[10] BISHOP, R.L. and GOLDBERG, S. I. : *On conformally flat spaces with commuting curvature and Ricci transformations*, Canad. J. Math., **XXIV**, No. 5 (1972), 799-804.

[11] ADATI, T. and MIYAZAWA, T. : *Some properties of P-Sasakian manifolds*, TRU Math., 13 (1977), 33-42.

[12] CHAKI, M.C. and GUPTA, B. : *On conformally symmetric spaces*, Indian J. Math., **5** (1963), 113-122.

[13] ADATI, T. and MATSUMOTO, K. : *On conformally recurrent and conformally symmetric P-Sasakian manifolds*, TRU Math., 13 (1977), 25-32.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI 741235
WEST BENGAL
INDIA

ÖZET

Bu çalışmada, C konform eğrilik tensörü, R Riemann eğriliği ve $R(X, Y)$ de tensör cebrinin, manifoldun her bir noktasında X, Y vektörleri için bir türev fonksiyonu olmak üzere, $R(X, Y) . C = 0$ koşulunu gerçekleyen bir tür P-Sasakian manifoldu incelenmektedir.