# A NOTE ON ABSOLUTE $\sigma$ - SUMMABILITY

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The purpose of this paper is to introduce and discuss the spaces of absolutely $\sigma$ - summable sequences.

## 1. INTRODUCTION

Let $S$ be the set of all sequences real or complex and let $l_\infty$, $c$ and $c_0$ denote, respectively the Banach spaces of bounded, convergent and null sequences normed as usual by $\|x\| = \sup_k |x_k|$.

Let $\sigma$ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) \neq n$ for all positive integers $n$ and $m$ where $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$, $m = 1, 2, \ldots$. A continuous linear functional $\phi$ on $l_\infty$, $c$ is an invariant mean or $\sigma$-mean if (i) $\phi(x) \geqslant 0$ when the sequence $x = (x_n)$ has $x_n \geqslant 0$ for all $n$, (ii) $\phi(e) = 1$, where $e = (1, 1, \ldots)$ and (iii) $\phi(x_{\sigma(n)}) = \phi(x)$, for all $x \in l_\infty$, when $\sigma(n) = n + 1$, a $\sigma$-mean is often called a Banach limit (see, Banach [1]) and $V_\sigma$, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences (see, Lorentz [2]).

If $x = (x_n)$, we write $Tx = (x_{\sigma(n)})$. The space $V_\sigma$ can be characterized either (i) as the set of all bounded sequences $x$ for which there is an $L$ so that $\lim_m t_{mn}(x) = L$ uniformly in $n$ where

$$t_{mn}(x) = \frac{1}{m+1}\left(\sum_{i=0}^{m} T^i(x)\right)_n \quad n = 1, 2, \ldots$$

(the $n$ th component of the sequence) or (ii) as the set of all bounded sequences $x$ for which $\lim_m \frac{1}{m+1} \sum_{i=0}^{m} T^i(x)$ is of the form $L\,e$, where $L = \sigma\text{-}\lim x$ (see, for explanation [5] and [9]).

Put

$$\Psi_{mn}(x) = t_{mn}(x) - t_{m-1,n}(x) .$$

A straightforward calculation shows that,

$$\Psi_{mn}(x) = \begin{cases} \dfrac{1}{m(m+1)} \displaystyle\sum_{j=1}^{m} j\,(T^j x_n - T^{j-1} x_n) , & (m \geqslant 1) \\ x_n , & (m = 0) . \end{cases}$$

Let $(p_m)$ be a sequence of real numbers such that $p_m > 0$ and $\sup p_m < \infty$. We define (see, Savaş [8]),

$$l^{\sigma}(p) = \left\{ x : \sum_m | \Psi_{mn}(x) |^{p_m} \text{ converges uniformly in } n \right\}$$

$$l^{\sigma\sigma}(p) = \left\{ x : \sup_n \sum_m | \Psi_{mn}(x) |^{p_m} < \infty \right\} .$$

If $p_m = p$ for all $m$ we write $l_p^{\sigma}$ and $l_p^{\sigma\sigma}$ in place of $l^{\sigma}(p)$ and $l^{\sigma\sigma}(p)$. If $p = 1$, we write $l^{\sigma}$ for $l_p^{\sigma}$ and this denotes the set of all absolutely $\sigma$-convergent sequences (see, Savaş [6]).

Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk} x_k$ converges for each $n$.

Let $X$ and $Y$ be any two nonempty subsets of $S$. If $x = (x_k) \in X$ then $Ax = (A_n(x)) \in Y$. We say that $A$ defines a matrix transformation from $X$ into $Y$ and we denote it by $A : X \longrightarrow Y$. By $(X, Y)$ we mean the class of matrices $A$ such that $A : X \longrightarrow Y$. If in $X$ and $Y$ there is some notion of limit or sum, then we write $(X, Y, P)$ to denote the subset of $(X, Y)$ which preserves the limit or sum.

The summability methods of real or complex sequences by infinite matrices are of three types (see, Maddox [3], p. 185) - ordinary, absolute and strong. In the same it is expected that the concept of invariant mean must give rise to three types of summability methods - invariant, absolute invariant and strongly invariant. The $\sigma$-summable sequences were introduced by Raimi [5] and discussed by Schaefer [9] and some others. The spaces of strongly $\sigma$-summable sequences have been discussed by Savaş [6, 7].

The purpose of this paper is to introduce and discuss the spaces of absolutely $\sigma$-summable sequences. Also some matrix transformations have been characterized.

## 2. ABSOLUTE $\sigma$ - SUMMABILITY

We define,

$$|A_\sigma\ , p| = \left\{ x : \sum_m |\Psi_{mn}(Ax)|^{p_m} \text{ converges uniformly in } n \right\}$$

$$|A_{\sigma\sigma}, p| = \left\{ x : \sup_n \sum_m |\Psi_{mn}(Ax)|^{p_m} < \infty \right\}$$

where

$$\Psi_{mn}(Ax) = \sum_k a(n, k, m)\, x_k$$

is such that

$$a(n, k, m) = \begin{cases} \dfrac{1}{m(m+1)} \displaystyle\sum_{j=1}^{m} j\left(a(\sigma^j(n), k) - a(\sigma^{j-1}(n), k)\right) & ; \quad m \geqslant 1 \\ a_{nk} & ; \quad m = 0. \end{cases}$$

The notation $a(n, k)$ denotes the element $a_{nk}$ of the matrix $A$. If $p_m = p$ for all $m$ we write $|A_\sigma|_p$ and $|A_{\sigma\sigma}|_p$ for $|A_\sigma, p|$ and $|A_{\sigma\sigma}, p|$ respectively.

When $\sigma(n) = n + 1$, $|A_\sigma, p|$ and $|A_{\sigma\sigma}, p|$ become $|\hat{A}, p|$ and $|\hat{\hat{A}}, p|$ which are studied in [4].

We have

**THEOREM 1.** $|A_\sigma, p| \subset |A_{\sigma\sigma}, p|$.

The proof follows on the same lines as adopted by Savaş [8] for $l^\sigma(p)$. So we omit it.

**REMARK.** It is now a pertinent question, whether $|A_{\sigma\sigma}, p| \subset |A_\sigma, p|$. We are not able to answer this question and it remains open.

A linear topological $X$ is called a paranormed space if there exists a subadditive function $g : X \longrightarrow R^+$ such that $g(0) = 0$, $g(x) = g(-x)$ and $\lambda \longrightarrow \lambda_0$, $x \longrightarrow x_0$ imply $\lambda x \longrightarrow \lambda_0 x_0$ for $\lambda \in C$ and $x \in X$

We now have

**THEOREM 2.** (i) $|A_\sigma, p|$ is a linear topological space paranormed by

$$g(x) = \sup_n \left( \sum_m \left| \sum_k a(n, k, m)\, x_k \right|^{p_m} \right)^{1/M} \tag{2.1}$$

where $M = \max(1, \sup p_m)$.

(ii) $|A_\sigma, p| \subset |A_\sigma, q|$ for $p_m \leqslant q_m$.

**Proof.** From Theorem 1, (2.1) is true for $x \in |A_\sigma, p|$. It can be proved by "standard" arguments that $g$ is a paranorm on $|A_\sigma, p|$. First, we will claim that for a fixed $x$, $\lambda x \longrightarrow 0$ as $\lambda \longrightarrow 0$. For, if $x \in |A_\sigma, p|$, then given $\varepsilon > 0$ there exists a $K$ such that, for all $n$,

$$\sum_{m \geq K} \left| \sum_k a(n, k, m) x_k \right|^{p_m} < \varepsilon . \tag{2.2}$$

So if $0 < \lambda \leqslant 1$, then

$$\sum_{m \geq K} \left| \sum_k \lambda a(n, k, m) x_k \right|^{p_m} \leqslant \sum_{m \geq K} \left| \sum_k a(n, k, m) x_k \right|^{p_m} < \varepsilon ,$$

and since for fixed $K$,

$$\sum_{m=0}^{K-1} \left| \sum_k \lambda a(n, k, m) x_k \right|^{p_m} \longrightarrow 0$$

as $\lambda \longrightarrow 0$.

If $p_m = p$ for all $m$, then $g$ is a norm for $p \geqslant 1$ and $p$-norm for $0 < p < 1$. To prove (ii), let $x \in |A_\sigma, p|$. Then there exists an integer $K$ such that

$$\sum_{m \geq K} \left| \sum_k a(n, k, m) x_k \right|^{p_m} \leqslant 1 .$$

Hence for $m \geqslant K$ $\left| \sum_k a(n, k, m) x_k \right| \leqslant 1$

and

$$\left| \sum_k a(n, k, m) x_k \right|^{q_m} \leqslant \left| \sum_k a(n, k, m) x_k \right|^{p_m} .$$

The uniform convergence of $\left| \sum_k a(n, k, m) x_k \right|^{p_m}$ therefore follows from that of $\left| \sum_k a(n, k, m) x_k \right|^{q_m}$.

**THEOREM 3.** Let $\inf p_m > 0$. Then $|A_{\sigma\sigma}, p|$ is a linear topological space paranormed by $g$.

**Proof.** The proof is routine, but there exists an essential difference between the proof of Theorem 3 and that of Theorem 2 (i). If $x \in |A_{\sigma\sigma}, p|$ then (2.2) is not true (by definition). We now use the assumption that $\inf p_m > 0$.

Let $\theta = \inf p_m$ such that $\theta > 0$. Then for $|\lambda| \leqslant 1$, $|\lambda|^{p_m} \leqslant |\lambda|^{\theta}$, so that $g(\lambda x) \leqslant |\lambda|^{\theta} g(x)$. The result now clearly follows.

## 3. MATRIX TRANSFORMATIONS

In this section we consider matrix transformations between some classes of sequences.

**THEOREM 4.** $A \in (l, l^{\sigma\sigma}, p)$ if and only if

$$\sup_{n, k} \sum_k |a(n, k, m)| < \infty \tag{3.1}$$

$$\sum_m a(n, k, m) = 1 \quad (\forall\ n, k). \tag{3.2}$$

The condition (3.1) is necessary and sufficient for $A \in (l, l^{\sigma\sigma})$ as mentioned in [8].

To prove the sufficiency of (3.2) we have

$$\sum_m \Psi_{mn}(A x) = \sum_m \sum_k a(n, k, m)\, x_k$$

$$= \sum_k x_k \sum_m a(n, k, m) = \sum_k x_k$$

the change of order of summation in the above step is justified by absolute convergence.

To prove the necessity of (3.2), suppose that $A \in (l, l^{\sigma\sigma}, P)$, that is, we are given that

$$\sum_m \Psi_{mn}\ (A x) = \sum_k x_k. \tag{3.3}$$

For a fixed $r \in Z^{+}$, define $x_k$ as

$$x_k = \begin{cases} 1, & k = r \\ 0, & k \neq r. \end{cases}$$

Then (3.3) reduces to condition (3.2) and by assumption $r$ is fixed the result follows.

Next we have

**THEOREM 5.** For $1 \leqslant p < \infty$, $(l, l_p^{\sigma\sigma})$ is a Banach space normed by

$$\| A \| = \sup_{n, m} \left( \sum_k |a(n, k, m)|^p \right)^{1/p}. \tag{3.4}$$

$(l, l^{\sigma\sigma}, P)$ is closed and convex in $(l, l^{\sigma\sigma})$. (3.5)

The proof uses ideas similar to those used in characterising $(l, l_p)$.

**REMARK.** The space $(c, V_\sigma)$ is a Banach space and $(c, V_\sigma, P)$ is closed and convex in $(c, V_\sigma)$. These results do not appear anywhere but can be proved as in Maddox in [3].

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## ÖZET

Bu çalışmada mutlak $\sigma$ - toplanabilir dizi uzayları incelenmektedir.