COMMUTATIVITY CONDITIONS FOR SEMI-PRIME RINGS *)

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In the present paper we prove the following : "Let m, n and s be any positive integers, and let R be a semi-prime ring with center $Z(R)$. If R satisfies one of the conditions $x^m y^n - x y^s x \in Z(R)$ or $x^m y^n - y x^s y \in Z(R)$ for each $x, y \in R$, then R is commutative". This result generalized Wei Zongxuan and Khan et al. Also the commutativity of a semi-prime ring has been proved under different set of conditions.

1. INTRODUCTION

Throughout the present paper R will represent an associative ring (may be without unit) with center $Z(R)$. Recently, Wei Zongxuan [5] proved that a semi-prime ring R is commutative if any one of the conditions

$$x^2 y^2 - x y^2 x \in Z(R) \text{ for all } x, y \in R, \tag{1}$$

$$x^2 y^2 - y x^2 y \in Z(R) \text{ for all } x, y \in R \tag{2}$$

is satisfied. Later, Khan et al [3] extended the above results. The purpose of the present paper is to generalize the results of Wei Zongxuan [5] and Khan et al [3]. Other commutativity theorems for semi-prime rings are obtained under different set of conditions. By $GF(q)$, we mean the Galois field (finite field) wit q elements, and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. Set $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $(GF(p))_2$ for a prime p.

2. PRELIMINARY RESULT

In preparation for the proof of our results, we need the following lemma:

Lemma ([2, Theorem]). Let f be a polynomial in n non-commuting

*) AMS (1980) Subject classification (1985 Revision) : 16 A 70.

Key words and phrases : Semi-prime ring, commutativity, polynomial identity.

indeterminates $x_1, x_2, \dots, x_n$ with relatively prime integral coefficients. Then the following are equivalent :

(1) For any ring satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal.

(2) For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.

(3) Every semi-prime ring satisfying $f = 0$ is commutative.

3. MAIN RESULTS

The following is the main results of this paper.

Theorem 1. Let m, n and s be any fixed positive integers, and let R be a semi-prime ring satisfying

$$x^m y^n - x y^s x \in Z(R) \quad \text{for all } x, y \in R. \tag{3}$$

Then R is commutative.

Proof. Let $x = e_{11}$ and $y = e_{12} + e_{22}$. Then

$$x^m y^n - x y^s x = e_{12} \notin Z(R).$$

Therefore, x and y fail to satisfy (3). Thus R is commutative by Lemma.

Theorem 2. Let m, n and s be any fixed positive integers, and let R be a semi-prime ring satisfying

$$x^m y^n - y x^s y \in Z(R) \quad \text{for all } x, y \in R. \tag{4}$$

Then R is commutative.

Proof. If $x = e_{11} + e_{12}$ and $y = e_{21} + e_{22}$, then

$$x^m y^n - y x^s y = e_{11} + e_{12} - e_{21} - e_{22} \notin Z(R).$$

Thus x and y fail to satisfy (4). Therefore, R is commutative by Lemma.

Theorem 3. Let m and n be fixed non-negative integers, and let R be a semi-prime ring satisfying

$$(yx)^m - x y^n x \in Z(R) \quad \text{for all } x, y \in R. \tag{5}$$

Then R is commutative.

Proof. Let $x = e_{12} + e_{21}$ and $y = e_{11}$. Now, if $(m, n) = (0, 0)$, then

$$x^2 = e_{11} + e_{22} \notin Z(R).$$

Let $m = 0$ and $n \geq 1$. Then (5) gives

$$x y^n x = e_{22} \notin Z(R).$$

Next, let $(m, n) = (1, 0)$. Then

$$yx = e_{12} \notin Z(R).$$

If $m = 1$ and $n \geq 1$, then (5) yields

$$yx - xy^n x = e_{12} - e_{22} \notin Z(R).$$

Finally, let $m > 1$. If $n = 0$, then

$$(yx)^m - x^2 = e_{11} + e_{22} \notin Z(R).$$

Suppose that $n \geq 1$. Then

$$(yx)^m - xy^n x = e_{22} \notin Z(R).$$

Hence in all cases x and y fail to satisfy (5). Therefore, R is commutative by Lemma.

Corollary 1 ([5, Theorem]). A semi-prime ring R is commutative if R satisfies one of the following conditions :

(1) $x^2 y^2 - xy^2 x \in Z(R)$ for every $x, y \in R$.

(2) $x^2 y^2 - yx^2 y \in Z(R)$ for every $x, y \in R$.

(3) $(yx)^2 - xy^2 x \in Z(R)$ for every $x, y \in R$.

Corollary 2 ([3, Theorem]). Let R be a semi-prime ring. Then the following statements are equivalent :

(1) R is commutative.

(2) There exist positive integers m and n such that

$$[x^m y^n - xy^n x, x] = 0 \quad \text{for all } x, y \in R.$$

(3) There exist positive integers s and t such that

$$[x^s y^t - yx^s y, x] = 0 \quad \text{for all } x, y \in R.$$

Example. The non-commutative ring of 3×3 strictly upper triangular matrices over the ring Z of integers which satisfies the conditions (2) and (3) of Corollary 2 rules out the possibility of extending the results of Theorem 1 and Theorem 2.

Theorem 4. Let m and n be fixed non-negative integers, and let R be a semi-prime ring. If R satisfies

$$(xy)^m - xy^n x \in Z(R) \quad \text{for all } x, y \in R, \tag{6}$$

then R is commutative.

Proof. In case $m \leq 1$, let $x = e_{12} + e_{21}$ and $y = e_{22}$. Suppose that $(m, n) = (0, 0)$. Then (6) gives

$$x^2 = e_{11} + e_{22} \notin Z(R).$$

If $m = 0$ and $n \geq 1$, then

$$x y^n x = e_{11} \notin Z(R).$$

Next, let $(m, n) = (1, 0)$. Then

$$x y = e_{12} \notin Z(R).$$

If $m = 1$ and $n \geq 1$, then

$$x y - x y^n x = - e_{11} + e_{12} \notin Z(R).$$

Now, let $m > 1$. Suppose that $x = e_{21} + e_{22}$ and $y = e_{22}$. Thus

$$(x y)^m - x y^n x = e_{22} - e_{21} - e_{22} = - e_{21} \notin Z(R).$$

By Lemma, R is commutative.

As a consequence of Theorem 4, we have

Corollary 3 ([4]). A semi-prime ring R satisfying

$$(x y)^2 - x y^2 x \in Z(R) \quad \text{for every } x, y \in R$$

is commutative.

Theorem 5. Let m and n be fixed non-negative integers, and let R be a semi-prime ring satisfying

$$x y^m x - y x^n y \in Z(R) \quad \text{for every } x, y \in R. \tag{7}$$

Then R is commutative.

Proof. Let $x = e_{22}$ and $y = e_{12} + e_{22}$. Then

$$x y^m x - y x^n y = e_{22} - e_{12} - e_{22} = - e_{12} \notin Z(R).$$

Thus x and y fail to satisfy (6). Therefore, R is commutative by Lemma.

Corollary 4 ([1]). If R is a semi-prime ring satisfying

$$x y^2 x - y x^2 y \in Z(R) \quad \text{for every } x, y \in R,$$

then R is commutative.

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ÖZET

Bu çalışmada Wei Zongxuan, Khan ve diğerlerinin elde etmiş oldukları sonuçları genelleştiren şu teorem ispat edilmektedir : "*m, n, s* herhangi üç pozitif tam sayı, *R* de merkezi $Z(R)$ olan bir yarı-asal halka olsun. Eğer *R* her $x, y \in R$ için $x^m y^n - x y^s x \in Z(R)$ ve $x^m y^n - y x^s y \in Z(R)$ koşullarından biri gerçekleniyorsa *R* komütatiftir". Bundan başka, farklı koşullar altında bir yarı-asal halkanın komütatifliği de ispat edilmektedir.