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ON THE CHARACTER TABLE OF THE WREATH PRODUCT 2.G Wr C_2

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In this paper we construct the character table of the extension 2. (G Wr C_2) if the character table of a finite group G is given. The group G Wr C_2 is the wreath product of G by the cyclic group C_2 and 2.G Wr C_2 is the extension of C_2 by the wreath product group G Wr C_2. Sometimes the wreath product G Wr C_2 is denoted by $G \wr C_2$ and the extension 2.G Wr C_2 is denoted by $C_2 \diagdown (G_1 C_2)$. Here we use some well known group-theoretic and character theoretic techniques to obtain some results that would be of great interest to some group theorists.

1. PRELIMINARIES

Let $H \trianglelefteq G$ (H Normal subgroup of G). A complement to H in G is a subgroup K of G with $G = HK$ and $H \cap K = \{1\}$. G is said to be an extension of a group X by a group Y if there exists $H \trianglelefteq G$ with $H \cong X$ (H isomorphic to X), and $G/H \cong Y$. The extension is said to split if H has a complement in G. The following construction can be used to describe split extensions.

Let S be the set product $A \times G$ and define a binary operation on S by $(a, g)\,(b, h) = (ab, g^{b\pi} h)$, $a, b \in A$, $g, h \in G$, where $g^{b\pi}$ denotes the image of G under the automorphism $b\pi$ of G, where $\pi : A \to \mathrm{Aut}(G)$ is a representation of A as a group of group automorphims of G.

We call S the semidirect product of G by A with respect to π. Denote S by $S(A, G, \pi)$, we have the following, see [1]:

(1.1) (a) $S = S(A, G, \pi)$ is a group.

(b) The maps $\sigma_A : A \to S$ and $\sigma_G : G \to S$ are injective group homomorphisms, where $\sigma_A : a \to (a, 1)$ and $\sigma_G : g \to (1, g)$.

(c) $G\sigma_G \trianglelefteq S$ and $A\sigma_A$ is a complement to $G\sigma_G$ in S.

(d) $(1, g)^{(a, 1)} = (1, g^{a\pi})$ for $g \in G$, $a \in A$.

(2.1) Let H be a group, $G \trianglelefteq H$, and B a complement to G in H. Let $\alpha : B \to \mathrm{Aut}(G)$ be the conjugation map (i. e. $b\alpha : g \to g^b$ for $b \in B$, $g \in G$). Define $\beta : S(B, G, \alpha) \to H$ by $(b, g)\,\beta = bg$, then β is an isomorphism.

We see from (1.1) and (2.1) that the semidirect products of G by A are precisely the split extensions of G by A. Moreover the representation defining the semidirect product is a conjugation map.

(3.1) Let L be a group and $\pi : G \to \mathrm{Sym}(X)$ a permutation representation of G on $X = \{1, \dots, n\}$. Form the direct product D of n-copies of L. G acts as a group of automorphisms of D via the representation α defined by

$$g\alpha : (x_1, \dots, x_n) \to (x_{1g^{-1}\pi}, \dots, x_{ng^{-1}\pi}).$$

(4.1) The wreath product of L by G (with respect to π) is defined to be the semidirect product $S(G, D, \alpha)$. The wreath product is denoted by $L \,\mathrm{Wr}\, G$ or $L \,\mathrm{Wr}_\pi\, G$ or $L\, Wr_x\, G$.

(5.1) Let $W = L\, Wr_\pi\, G$ be the wreath product of L by G with respect to π. Then we have the following theorem [1]:

(a) W is a semidirect product of D by G where $D = L_1 \times L_2 \times \dots \times L_n$ is a direct product of n copies of L.

(b) G permutes $\Delta = \{L_i : 1 \leqslant i \leqslant n\}$ via conjugation and the permutation representation of G on Δ is equivalent to π, that is $(L_i)^g = L_{ig\pi}$ for each $g \in G$ and $1 \leqslant i \leqslant n$.

Remark. For the wreath product G Wr C_2, let $\pi : C_2 \to \mathrm{Sym}(X)$ be a permutation representation of C_2 on $X = \{1, 2\}$. Form the direct product D of two copies of G. C_2 acts as a group aut of omorphisms of D via the representation α defined by $h\alpha : (x_1, x_2) \to (x_{1h^{-1}\pi}, x_{2h^{-1}\pi})$, $h \in C_2$. The wreath product of G by C_2 is the semidirect product $S(C_2, D, \alpha)$ which is sometimes denoted by $D : C_2$ or $D \searrow C_2$.

2. Lemma. Let m be the number of the conjugacy classes of G, then the number of the conjugacy classes of G Wr C_2 is $\dfrac{m(m+3)}{2}$.

Proof. Let $\overline{f_1}, \overline{f_2}, \dots, \overline{f_m}$ be the conjugacy classes of G, where $f_1, f_2, \dots, f_m$ are the representatives of these classes respectively. From (5.1) and from the above remark, the wreath product G Wr C_2 is isomorphic to $(G \times G) : C_2$. Let $\begin{bmatrix} \overline{f_\lambda} & \\ & \overline{f_\xi} \end{bmatrix}$ be a conjugacy class of $G \times G$ and let $C_2 = \left\langle \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right\rangle$ then $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \overline{f_\lambda} & \\ & \overline{f_\xi} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} \overline{f_\xi} & \\ & \overline{f_\lambda} \end{bmatrix}$, so, the class of the type $\begin{bmatrix} \overline{f_\lambda} & \\ & \overline{f_\xi} \end{bmatrix}$

in $G \times G$ conjugates with $\begin{bmatrix} \bar{f}_\xi & \\ & \bar{f}_\lambda \end{bmatrix}$ to give the conjugacy class $\left(\begin{bmatrix} \bar{f}_\lambda & \\ & \bar{f}_\xi \end{bmatrix}, \begin{bmatrix} \bar{f}_\xi & \\ & \bar{f}_\lambda \end{bmatrix} \right)$ of $G \times G : C_2$ and hence the number of the conjugacy classes of this form is $\frac{m!}{(m-2)!2!} + m$, and there are m conjugacy classes of the form $\begin{bmatrix} \bar{f}_\eta & \\ & \bar{f}_\nu \end{bmatrix}$ do not fuse with any other class, and thus total number of the conjugacy classes of $G \times G : C_2$ is $\frac{m!}{(m-2)!2} + 2m = \frac{m(m+3)}{2}$.

3. THE IRREDUCIBLE CHARACTERS OF $G \times G : C_2$

Let $\psi_1, \psi_2, \ldots, \psi_m$ be the irreducible characters of G, then we have m^2 irreducible characters of $G \times G$ of the form $\psi_i \times \psi_j$, $m \leqslant i \leqslant 1$, $m \leqslant j \leqslant 1$. The irreducible characters $\psi_i \times \psi_j$ $(i \neq j)$ of $G \times G$ fuse in pairs (i.e. $\psi_i \times \psi_j$ fuses with $\psi_j \times \psi_i$) to give $\frac{m!}{(m-2)!2}$ irreducible characters of $G \times G : C_2$. Let $\overline{\psi_i \times \psi_j}$ denote the irreducible character of $G \times G : G_2$ resulting from the fusion of $\psi_i \times \psi_j$ with $\psi_j \times \psi_i$, $(i \neq j)$. The irreducible character $\overline{\psi_i \times \psi_j}$ has zero values outside $G \times G$ and its values on $G \times G$ are the same as of the values of $\psi_i \times \psi_j$ on $G \times G$. The irreducible characters $\psi_\alpha \times \psi_\alpha$ of $G \times G$ split into two irreducible characters $\overline{(\psi_\alpha \times \psi_\alpha)_1}$ and $\overline{(\psi_\alpha \times \psi_\alpha)_2}$ of $G \times G : C_2$. Their values outside $G \times G$ are the same as values of ψ_α and $-\psi_\alpha$ on G respectively, this can be proved as follows:

Proposition. (3.1) Let ψ be an irreducible character of G, then $\psi \times \psi$ is an irreducible character of $G \times G$. We extend $\psi \times \psi$ to an irreducible character $\overline{\psi \times \psi}$ of $G \times G : C_2$.

Let $N = \{(g, g), g \in G\}$, then N is a subgroup of $G \times G$ and $N \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \simeq G \times C_2$. The restriction of $\overline{\psi \times \psi}$ to $G \times C_2$ gives the irreducible character $\psi \times \theta$ where θ is an irreducible character of C_2. The values of $\overline{(\psi_\alpha \times \psi_\alpha)_1}$ and $\overline{(\psi_\alpha \times \psi_\alpha)_2}$ on $G \times G$ are the same as the values of $\psi_\alpha \times \psi_\alpha$ on $G \times G$.

Note. The centralizer of elements of the conjugacy classes of $G \times G : C_2$ which fuse does not change, but the centralizer of the elements of the conjugacy classes which does not fuse is doubled. In Table 3.2 we exhibit the general form of the character table of $G \times G : C_2$. In this table, our notation for conjugacy classes, characters and so on follows the Atlas [2], for instance the characters which fuse are joined by a fusion symbol ($\int$), and the characters which split are indicated by the symbol (:). Also $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is denoted by the letter e. The conjugacy classes of $G \times G$ are denoted by $\left(\begin{bmatrix} \overline{f_i} & \\ & \overline{f_j} \end{bmatrix}, \begin{bmatrix} \overline{f_i} & \\ & \overline{f_i} \end{bmatrix} \right)$ and $C_1, C_2, \ldots, C_m$ denote the centralizers of the representatives $f_1, f_2, \ldots, f_m$ of the conjugacy classes $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_m}$.

4. THE CHARACTER TABLE OF $2.G \times G : C_2$

Since $2.G \times G : C_2 \cong G \times G : C_2 / \langle -1 \rangle$ then its character table can be easily obtained using the following lemma:

Lemma 4.1 [3]. Let $E \trianglelefteq H$.

(a) If χ is a character of H and $E \subseteq \mathrm{Ker}\, \chi$, then χ is constant on cosets E in H and the function $\hat{\chi}$ on H/E defined by $\hat{\chi}(Eh) = \chi(h)$ is a character of H/E.

(b) If $\hat{\chi}$ is a character of H/E, then the function χ defined by $\chi(h) = \hat{\chi}(Eh)$ is a character of H.

(c) In both (a) and (b) χ is an irreducible character of H iff $\hat{\chi}$ is an irreducible character of H/E.

Table 3.2

Conjugacy Classes	$\left(\begin{bmatrix}\bar{f}_1 & \\ & \bar{f}_1\end{bmatrix}\right)$,	$\left(\begin{bmatrix}\bar{f}_1 & \\ & \bar{f}_2\end{bmatrix}, \begin{bmatrix}\bar{f}_2 & \\ & \bar{f}_1\end{bmatrix}\right)$	$, \ldots,$	$\left(\begin{bmatrix}\bar{f}_m & \\ & \bar{f}_1\end{bmatrix}, \begin{bmatrix}\bar{f}_1 & \\ & \bar{f}_m\end{bmatrix}\right)$	$, \ldots,$	$\left(\begin{bmatrix}\bar{f}_m & \\ & \bar{f}_m\end{bmatrix}\right)$	$: e\bar{f}_1, \ldots, e\bar{f}_m$
Centralizers	$2\,C_1^2$	$C_1\,C_2$	$, \ldots,$	$C_m\,C_1$	$, \ldots,$	$2\,C_m^2$	$: 2\,C_1, \ldots, 2\,C_m$
Characters	$\psi_1 \times \psi_2$	$\Big\| \Rightarrow \overline{\psi_1 \times \psi_2}$				$\int$	
	$\psi_2 \times \psi_1$						
	$\vdots$					$\vdots$	
$\psi_i \times \psi_j$	$\psi_1 \times \psi_m$	$\Big\| \Rightarrow \overline{\psi_1 \times \psi_m}$				$\int$	0
$\psi_j \times \psi_i$	$\psi_m \times \psi_1$					$\int$	
$i \neq j$	$\vdots$					$\vdots$	
$m \leqslant i \leqslant 1$	$\psi_m \times \psi_j$	$\Big\| \Rightarrow \overline{\psi_m \times \psi_j}$				$\int$	
$m \leqslant j \leqslant 1$	$\psi_j \times \psi_m$					$\vdots$	
	$\vdots$					$\int$	
$\psi_i \times \psi_i$	$\psi_1 \times \psi_1$					$:$	$\pm\,\psi_1$
$i = 1, 2, \ldots, m$	$\psi_2 \times \psi_2$					$:$	$\pm\,\psi_2$
	$\vdots$					$\vdots$	$\vdots$
	$\psi_m \times \psi_m$					$:$	$\pm\,\psi_m$

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ÖZET

Bu çalışmada, sonlu bir G grubunun karakter tablosunun verilmesi halinde, 2.(G Wr C_2) genişlemesinin karakter tablosu inşa edilmekte ve bu arada, gruplar teorisi konusunda çalışan bazı matematikçiler için çok ilginç olabilecek sonuçlar elde etmek için, gruplar teorisi ve karakter teorisi ile ilgili, çok iyi bilinen bazı teknikler kullanılmaktadır.