SUBCLASSES OF UNIVALENT MEROMORPHIC FUNCTIONS

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Let M_n be the classes of regular functions $f(z)=z^{-1}+a_0+a_1 z+\dots$ defined in the annulus $0<|z|<1$ and satisfying $\operatorname{Re} \dfrac{I^{n+1} f(z)}{I^n f(z)} > 0$ ($n=0, 1, 2, \dots$), where $I f(z)=f(z)*(z^{-1}-z(1-z)^{-2})$, $I^n f(z)=I(I^{n-1} f(z))$ and $*$ is the Hadamard convolution. We denote by $R_n(\alpha, \beta)$ the set of all functions $f(z)=z^{-1}+a_0+a_1 z+\dots$ such that

$$\operatorname{Re}\left\{(\alpha+\beta)\frac{I^{n+1} f(z)}{I^n f(z)}-\alpha\frac{I^{n+2} f(z)}{I^{n+1} f(z)}\right\}>0 \qquad (|z|<1, \alpha>0, \beta>0).$$

It is proved that $M_{n+1} \subset M_n$ and $R_n(\alpha, \beta) \subset M_n$. In particular we obtain the radius of M_{n+1} for the class M_n. Further we consider the integrals of functions in M_n.

1. INTRODUCTION

Let Σ denote the family of functions of the form $f(z)=\dfrac{1}{z}+a_0+a_1 z+\dots$ regular in the annulus $0<|z|<1$. The Hadamard product or the convolution of two functions $f, g \in \Sigma$ will be denoted by $f*g$. The convolution has algebraic properties of ordinary multiplication.

Let $n \in N_0=\{0, 1, 2, 3, \dots\}$. We define a linear operator I^n on Σ by

$$I^0 f(z)=f(z)$$

$$I^1 f(z)=I f(z)=-z f'(z)$$

$$I^n f(z)=I(I^{n-1} f(z))=-z(I^{n-1} f(z))'.$$

This operator is an iterated convolution [5]. If $h(z)=\dfrac{1}{z}-\dfrac{z}{(1-z)^2}$, then

$$I^n f(z)=((h*h*\dots*h)*f)(z)=\frac{1}{z}+(-1)^n \sum_{k=1}^{\infty} k^n a_k z^k.$$

Now we introduce the following classes: Let E denote the unit disc, $\{z:|z|<1\}$. We define the classes M_n of funtions $f \in \Sigma$ and satisfying the condition

$$\operatorname{Re} \frac{I^{n+1} f(z)}{I^n f(z)} > 0 \quad (z \in E, n \in N_0).$$

For every $n \in N_0$, M_n contains many interesting classes of univalent functions: M_0 and M_1 are known classes of univalent functions that are meromorphicaliy starlike and convex respectively. Let

$$r_n(z) = (\alpha + \beta) \frac{I^{n+1} f(z)}{I^n f(z)} - \alpha \frac{I^{n+2} f(z)}{I^{n+1} f(z)} \tag{1.1}$$

where $n \in N_0$ and α, β are non-negative real numbers. We say that $f \in R_n(\alpha, \beta)$, if $f \in \Sigma$ and

$$\operatorname{Re} r_n(z) > 0 \quad (z \in E, \ \alpha > 0, \ \beta > 0, \ n \in N_0).$$

It is clear that $R_n(0, \beta) = M_n$, $R_n(-1, 1) = M_{n+1}$.

In Section 2 we shall show that

$$M_{n+1} \subset M_n \quad (n \in N_0). \tag{1.2}$$

Methods used are similar to those in [4]. Since M_0 equals Σ^* (the class of meromorphically starlike functions) the univalence of members in M_n is a consequence of (1.2). In particular we obtain the radius of M_{n+1} for the class M_n.

Note that when $n = 0$, this number $2 - \sqrt{3}$ is called the radius of convexity for the class of meromorphicaliy starlike functions. Next we shall show that $R_n(\alpha, \beta) \subset M_n$. For $n = 0$ it follows that $R_0(\alpha, \beta) \subset \Sigma^*$. This result is a generalization of the result obtained by Bajpai-Mehrok in [1]. In Section 3 we study special elements of M_n which have certain integral representations. Our results are thus generalizations of the results obtained by Goel-Sohi in [2].

2. THE CLASSES M_n AND $R_n(\alpha, \beta)$

Theorem 1. $M_{n+1} \subset M_n$ for all $n \in N_0$.

Proof. Let $f \in M_{n+1}$. We define $w(z)$ in E by

$$\frac{I^{n+1} f(z)}{I^n f(z)} = \frac{1 - w(z)}{1 + w(z)}. \tag{2.1}$$

Here $w(z)$ is a regular function in E with $w(0) = 0$ and $w(z) \neq -1$ for $z \in E$. Differentiating (2.1) logarithmically we obtain

$$\frac{I^{n+2} f(z)}{I^{n+1} f(z)} = \frac{1 - w(z)}{1 + w(z)} + \frac{2 z w'(z)}{(1 - w(z))(1 + w(z))}. \tag{2.2}$$

Equation (2.1) should yield $|w(z)| < 1$ for all $z \in E$, otherwise by a lemma of Jack [3] there exists $z_0 \in E$ such that $z_0 w'(z_0) = m w(z_0)$, $m \geqq 1$, $|w(z_0)| = 1$. Applying this result to (2.2) we get

$$\operatorname{Re}\frac{I^{n+2}f(z_0)}{I^{n+1}f(z_0)}=0.$$

This is a contradiction to the assumption that $f\in M_{n+1}$. Hence $f\in M_n$ when $n\in N_0$.

Remark. Since M_0 equals Σ^* (the class of starlike functions), it follows from the above theorem that all functions in M_n are univalent.

Theorem 2. Let $f\in M_n$. Then $\operatorname{Re}\dfrac{I^{n+2}f(z)}{I^{n+1}f(z)}>0$ holds for $|z|<2-\sqrt{3}$.

Proof. Let $p(z)$ be the regular function defined in E by

$$\frac{I^{n+1}f(z)}{I^{n}f(z)}=p(z). \tag{2.3}$$

Here $p(0)=1$ and $\operatorname{Re}p(z)>0$ in E. Logarithmic differentiation of (2.3) yields

$$\frac{I^{n+2}f(z)}{I^{n+1}f(z)}=p(z)-\frac{zp'(z)}{p(z)}. \tag{2.4}$$

Using the well-known estimates $\left|\dfrac{zp'(z)}{p(z)}\right|\leqq\dfrac{2|z|}{1-|z|}$ and $\operatorname{Re}p(z)\geqq\dfrac{1-|z|}{1+|z|}$,

we see from (2.4) that

$$\operatorname{Re}\frac{I^{n+2}f(z)}{I^{n+1}f(z)}\geqq\operatorname{Re}p(z)\,.\,\frac{|z|^2-4|z|+1}{1-|z|^2}. \tag{2.5}$$

Now the right hand side of (2.5) is positive provided $|z|<2-\sqrt{3}$.

Theorem 3. $R_n(\alpha,\beta)\subset M_n$ for all $n\in N$, $\alpha>0$ and $\beta>0$.

Proof. Suppose $f\in R_n(\alpha,\beta)$ and

$$\frac{I^{n+1}f(z)}{I^{n}f(z)}=\frac{1-w(z)}{1+w(z)}\qquad(z\in E). \tag{2.6}$$

Then $w(z)$ is regular in E with $w(0)=0$, $w(z)\neq-1$. To complete the proof we need to show that $\operatorname{Re}\dfrac{1-w(z)}{1+w(z)}>0$, $z\in E$. Taking the logarithmic derivative of both sides of (2.6) we get

$$\frac{I^{n+2}f(z)}{I^{n+1}f(z)}=\frac{1-w(z)}{1+w(z)}+\frac{2zw'(z)}{(1-w(z))(1+w(z))}. \tag{2.7}$$

Substituting from (2.6) and (2.7) in (1.1) we obtain

$$r_n(z) = \beta \frac{1 - w(z)}{1 + w(z)} - \alpha \frac{2 z w'(z)}{(1 - w(z))(1 + w(z))}. \tag{2.8}$$

The conclusion of the theorem from (2.8) follows as in Theorem 1.

3. SPECIAL ELEMENTS OF M_n

Theorem 4. Let $n \in N_0$ and $c > 0$. If $f \in M_n$, then

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\, dt \tag{3.1}$$

also belongs to M_n for $F(z) \neq 0$ in $0 < |z| < 1$.

Proof. Since

$$z F'(z) = c f(z) - (c + 1) F(z)$$

it can be easily verified that

$$I^{n+1} F(z) = (c + 1) I^n F(z) - c I^n f(z). \tag{3.2}$$

Let $w(z)$ be the regular function in E defined by

$$\frac{I^{n+1} F(z)}{I^n F(z)} = \frac{1 - w(z)}{1 + w(z)} \quad (z \in E). \tag{3.3}$$

Obviously $w(0) = 0$, $w(z) \neq -1$ for $z \in E$. Logarithmic derivative (3.3) and using (3.2) we obtain

$$\frac{I^{n+1} f(z)}{I^n f(z)} = \frac{1 - w(z)}{1 + w(z)} - \frac{2 z w'(z)}{(1 + w(z))(c + (c + 2) w(z))}. \tag{3.4}$$

We claim that $|w(z)| < 1$ in E, otherwise by a lemma of Jack [3] there exists $z_0 \in E$ such that $z_0 w'(z_0) = m w(z_0)$, $m \geqq 1$, $|w(z_0)| = 1$. Thus at $z = z_0$, from (3.4) we see that

$$\operatorname{Re} \frac{I^{n+1} f(z_0)}{I^n f(z_0)} < 0.$$

This completes the proof of the theorem by contradiction.

We shall use the following lemma due to Goel-Sohi ([2], Theorem 4):

Lemma. If $w(z)$ is regular in E and satisfies the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in E$, then

$$\operatorname{Re} \left\{ \frac{1 - w(z)}{1 + w(z)} - \frac{2 z w'(z)}{(1 + w(z))(c + (c + 2) w(z))} \right\} > 0$$

for $|z| < \sqrt{\frac{c}{c+2}}$. The result is sharp.

Theorem 5. Let $F \in M_n$, $c>0$, and $f(z) = \frac{1}{c z^c} (z^{c+1} F(z))'$, then $f \in M_n$ for $0 < |z| < \sqrt{\frac{c}{c+2}}$. The result is sharp.

Proof. Since $F \in M_n$, there exists a function $w(z)$ regular in E with $w(0)=0$, $|w(z)| < 1$ such that

$$\frac{I^{n+1} F(z)}{I^n F(z)} = \frac{1-w(z)}{1+w(z)} .$$

We find (3.4) from the Theorem 4. It follows that from lemma

$$\operatorname{Re} \frac{I^{n+1} f(z)}{I^n f(z)} > 0$$

for $0 < |z| < \sqrt{\frac{c}{c+2}}$.

Let $F(z) = \frac{1}{z} + 2 + z$ and $z_0 = \sqrt{\frac{c}{c+2}}$. Then $I^n F(z) = \frac{1}{z} + (-1)^n . z$ and

$$\operatorname{Re} \frac{I^{n+1} F(z)}{I^n F(z)} = \operatorname{Re} \frac{1-\varepsilon b z}{1+\varepsilon b z^2} > 0 \left(\varepsilon = (-1)^n ,\ b = \frac{c+2}{c}\right).$$

Thus, $F \in M_n$. From (3.5) we obtain $f(z) = \frac{1}{z} + 2 \frac{c+1}{c} + \frac{c+2}{c} z$,

$I^n f(z) = \frac{1}{z} + \varepsilon b z \left(\varepsilon = (-1)^n ,\ b = \frac{c+2}{c}\right)$ and

$$\operatorname{Re} \frac{I^{n+1} f(z)}{I^n f(z)} = \operatorname{Re} \frac{1-\varepsilon b z^2}{1+\varepsilon b z^2} > 0 \quad (z \in E).$$

Thus, $f \in M_n$. Here $f'(z_0) = 0$ and $\operatorname{Re} \frac{I^{n+1} f(z_0)}{I^n f(z_0)} = 0$. Also, the result is sharp for the function $F(z) = \frac{1}{z} + 2 + z$.

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ÖZET

Bu makalede üstelenmiş Hadamard çarpımı yardımıyla tanımlanan sınıfların yalmkatiığı gösterilmektedir. Ayrıca fonksiyonların integral dönüşümleri incelenmektedir.