AN EXTREMAL PROBLEM FOR UNIVALENT FUNCTIONS *)

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In this paper, we determine the variability region of $R(F)$ in terms of the elliptic modular function. Here, $R(F)$ denotes the cross ratio of the images of four distinct points given in $|z| > 1$ and the function F varies in Σ.

1. Preliminary remarks. Let D be a simply connected domain in the closed (extended) complex plane C. For given distinct points $z_k (k = 1, 2, 3, 4)$ in D, let (z_1, z_2, z_3, z_4) denote their cross ratio. It is well known that if $w = h(z)$ is a homography and $w_k = h(z_k)$, then $(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$, but it may not be so if the homography h has been replaced by an arbitrary univalent function f in D. If $w = f(z)$ is univalent in D then the quotient $(w_1, w_2, w_3, w_4) / (z_1, z_2, z_3, z_4)$ denoted by Q determines a "measure of deviation" of $f(z)$ from the homography.

It is also interesting to observe that in some specific cases the quantity Q reduces to some well-known functionals (for example, $f(z)$, $f(z_1)/f(z_2)$, $zf'(z)/f(z)$) over some classes of univalent functions in or outside of the unit disc.

Motivated by these, we address ourselves to determination of the variability region of the cross ratio $(f(z_1), f(z_2), f(z_3), f(z_4))$ over the class of meromorphic and univalent functions in the complement of the unit disc. Our solution is based on the method of the Schiffer Boundary Variations and is defined in terms of elliptic modular function and hyperelliptic integrals.

2. The cross ratio problem. We shall give here some necessary definitions, notations and we shall state the problem.

Let $\Delta' = \{ z \in C : |z| > 1 \}$ and let Σ denote the class of all functions $F(z) = z + a_0 + \frac{a_1}{z} + \dots$, meromorphic and univalent in Δ'. Let Σ' be the subclass of Σ consisting of all functions $F(z)$ subject to the condition $a_0 = 0$, $z \in \Delta'$. If $z_k (k = 1, 2, 3, 4)$ are given distinct points in Δ' and $w_k = F(z_k)$ with $F \in \Sigma$, then the quantity

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$$R(F) = (w_1, w_2, w_3, w_4) = \frac{w_3 - w_1}{w_3 - w_2} \cdot \frac{w_4 - w_2}{w_4 - w_1}$$

is said to be the cross ratio of the four points. We define

$$E = \{ p : p = R(F) \text{ and } F \in \Sigma \}. \tag{1}$$

Since $R(F)$ is not altered by translations, there is no loss of generality in assuming that $F \in \Sigma'$. Now, the class Σ' is compact and connected, and the transformation $F \to R(F)$ is continuous on Σ'. Hence, the set F is compact and connected.

Let ∂E denote the boundary of E. We recall that a point $P_0 \in E$ is said to be the regular boundary point of E if there exists a point a, $a \in \mathbf{C} \setminus E$ such that for a certain disc $K(a, \varepsilon)$, $\varepsilon > 0$ there holds

$$\overline{K(a, \varepsilon)} \cap \partial E = \{ P_0 \}. \tag{2}$$

It is known [4] that regular boundary points form a dense subset of ∂E. Hence, in order to find the set E it suffices to determine all its regular boundary points. We shall now be concerned with the following problem:

For given distinct points z_k $(k = 1, 2, 3, 4)$ in Δ' and for F running over the whole class Σ', find the set E, the variability region of $R(F)$.

3. The necessary condition. We shall derive here a differential equation satisfied by functions that contribute regular boundary points of the variability region. We shall call such functions extremal.

Let P_0 be a regular boundary point of E, then there exists a function F_0 in Σ' such that $P_0 = R(F_0)$. The extremal function maps Δ' outside the complement of a continuum, say, K. Let w_0 be a point in K. By Schiffer's Theorem (1, p. 297) there exist functions in Σ' whose linear parts are given by

$$w^* = w + \frac{\lambda(r)}{w - w_0} + o(\lambda(r)), \; w = F_0(z).$$

Now

$$w_k^* = w_k + \frac{\lambda(r)}{w_k - w_0} + o(\lambda(r)),$$

and we find

$$R(w^*) = R(w) + \frac{A \lambda(r)}{(w_0 - w_1)(w_0 - w_2)(w_0 - w_3)(w_0 - w_4)} + o(\lambda(r)),$$

where A depends upon w_k $(k = 1, 2, 3, 4)$, but the explicit dependence is of no importance. Setting $P_0 - a = | P_0 - a | e^{it}$ and making use of (2) we find

$$| R(w^*) - a|^2 = | R(w) - a|^2 + \\ + 2| R(w)-a| \operatorname{Re} \left\{ e^{-it} \frac{A \lambda (r)}{(w_0-w_1)(w_0-w_2)(w_0-w_3)(w_0-w_4)} \right\} + o(\lambda(r))$$

which implies the condition

$$\operatorname{Re} \left\{ e^{-it} \frac{A \lambda (r)}{(w_0-w_1)(w_0-w_2)(w_0-w_3)(w_0-w_4)} + o(\lambda(r)) \right\} \geqslant 0 .$$

By Schiffer's Theorem cited above it follows that the set $\mathbf{C} \setminus F_0(\Delta')$ is a finite system of analytic arcs each of which satisfies the condition

$$e^{-it} A \frac{(dw)^2}{(w-w_1)(w-w_2)(w-w_3)(w-w_4)} \leqslant 0. \tag{3}$$

The set $F_0(|z|=1)$ lies on a trajectory of the quadratic differential (3). This differential has four simple poles and no zeros. It follows that $F_0(|z|=1)$ is a single analytic arc and that F_0 is analytic on the unit circumference. Applyir g the reflection principle to the function in (3), we conclude that $w=F_0(z)$ must satisfy the following equation

$$\frac{e^{-it} A (dw)^2}{(w-w_1)(w-w_2)(w-w_3)(w-w_4)} = \frac{B(z-e^{i\alpha})^2 (z-e^{i\beta})^2 (dz)^2}{(z-z_1)(1-\bar{z}_1 z)\dots(z-z_4)(1-\bar{z}_4 z)}, \tag{4}$$

where B is a constant while $e^{i\alpha}$, $e^{i\beta}$ are the points on $|z|=1$ which are carried by F_0 onto the end-points of the analytic arc. We have necessarily $F_0'(e^{i\alpha}) = F_0'(e^{i\beta}) = 0$.

4. **Relation between parameters.** We shall show here that there is exactly one essential parameter in (4).

By multiplying both sides of (4) by z^4 and then letting $z \to \infty$, we obtain

$$A e^{-it} \overline{(z_1 z_2 z_3 z_4)} = B. \tag{5}$$

Secondly, the right hand side of (4) is non-positive on $|z|=1$. With $B=|B| e^{i\gamma}$, it gives (on $|z|=1$)

$$e^{i\gamma} \left(z + \frac{e^{i\alpha} e^{i\beta}}{z} - (e^{i\alpha} + e^{i\beta}) \right)^2 \geqslant 0 ,$$

or equivalently

$$e^{i(\alpha+\beta)} = e^{-i\gamma}. \tag{6}$$

Let $-Q(z)$ stand for the right hand side of (4). We have noticed that $F_0(|z|=1)$ is an analytic arc. Hence, there are two arcs on $|z|=1$ with common end points $e^{i\alpha}$, $e^{i\beta}$ which are carried over by F_0 onto opposite edges of the slit. Hence, we have [4], [3]

$$\int_{\alpha}^{\beta} \sqrt{Q(e^{i\theta})}\, d\theta = \int_{\beta}^{\alpha+2\pi} \sqrt{Q(e^{i\theta})}\, d\theta .$$

After some calculus this gives

$$\int_{\alpha}^{\alpha+2\pi} P(\theta) \sin \frac{\theta-\alpha}{2} \sin \frac{\theta-\beta}{2}\, d\theta = 0 ,$$

where $P(\theta) = |(e^{i\theta}-z_1)(e^{i\theta}-z_2)(e^{i\theta}-z_3)(e^{i\theta}-z_4)|^{-1/2}$. Since the integrand is a 2π - periodic function we obtain

$$\int_{0}^{2\pi} P(\theta) \sin \frac{\theta-\alpha}{2} \sin \frac{\theta-\beta}{2}\, d\theta = 0 ,$$

or ultimately

$$\int_{|z|=1} \frac{(z-e^{i\alpha})(z-e^{i\beta})}{\sqrt{(z-z_1)(1-\bar{z}_1 z)\dots(z-z_4)(1-\bar{z}_4 z)}}\, dz = 0 . \qquad (7)$$

Given $e^{i\alpha}$, one can find now a unique $e^{i\beta}$. The relations (5), (6) and (7) allow us to take $e^{i\alpha}$ as the only parameter related to regular boundary points and the equation (4) takes ultimately the form

$$\frac{(dw)^2}{(w-w_1)(w-w_2)(w-w_3)(w-w_4)} = \frac{\bar{z}_1 \bar{z}_2 \bar{z}_3 \bar{z}_4 (z-e^{i\alpha})^2 (z-e^{i\beta})^2 (dz)^2}{(z-z_1)(1-\bar{z}_1 z)\dots(z-z_4)(1-\bar{z}_4 z)} . \qquad (8)$$

For further properties let us write (8) in the form

$$P(w)\, dw^2 = Q(z)\, dz^2 . \qquad (9)$$

5. The variability region. Here we give a parametric equation satisfied by all regular boundary points of E. The form of the solution implies that they are the only regular boundary points.

We know that (9) has a single valued univalent solution $w = w(z)$ in the domain Δ'. If τ is a path situated in Δ' that starts from $z = \infty$, and if $\Gamma = w(\tau)$, then we have

$$\int_{\Gamma} \sqrt{P(w)}\, dw = \int_{\tau} \sqrt{Q(z)}\, dz . \qquad (10)$$

The condition (7) implies that the integral of $\sqrt{Q(z)}$ along any loop homotopic to $|z| = 1$ with respect to the domain $\Delta' - \{z_1, z_2, z_3, z_4\}$ equals zero. Hence, the integral $\int_{\tau} Q(z)\, dz$ is a hyperelliptic integral which has two primitive periods defined by loops in Δ' that surround two critical points of $Q(z)$ and that leave two remaining critical points outside. Similarly, the integral $\int_{\Gamma} P(w)\,dw$ is an elliptic integral and from its form it follows that the integrals

$$\int_{w_1}^{w_2} \sqrt{P(w)}\,dw\,, \quad \int_{w_1}^{w_3} \sqrt{P(w)}\,dw$$

are the primitive periods. The condition of single valuedness implies that we have

$$\frac{\int_{w_1}^{w_3} \sqrt{P(w)}\,dw}{\int_{w_1}^{w_2} \sqrt{P(w)}\,dw} = \frac{\int_{z_1}^{z_3} \sqrt{Q(z)}\,dz}{\int_{z_1}^{z_2} \sqrt{Q(z)}\,dz}\,. \tag{11}$$

We can now perform the change of variables along the formula

$$x = \frac{w-a}{w-w_4} \cdot \frac{w_1-w_4}{w_1-a} \quad (a = w_2,\ w_3 \text{ respectively}),$$

which brings the left hand side of (11) to the form $K(\rho)\,/\,K(1-\rho)$ where $\rho = \{(w_3-w_1)(w_4-w_2)\}\,/\,\{(w_3-w_2)(w_4-w_1)\}$ and $K(\rho) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-\rho \sin^2 x}}$. It is known, however (see [3], chapter 6), that the function

$$z = f(w) = K(1-w)\,/\,K(w)\,,$$

($f(w)$ being positive if $0 < w < 1$) is the inverse of the elliptic modular function which is commonly denoted by $\lambda(z)$. So ultimately, we arrive at the formula

$$\rho = R(F) = (w_1, w_2, w_3, w_4) = \lambda \left[\frac{\int_{z_1}^{z_2} \sqrt{Q(z)}\,dz}{\int_{z_1}^{z_3} \sqrt{Q(z)}\,dz} \right], \tag{12}$$

since the quotient may be formed in such a way that it has positive imaginary part.

In view of (6) and (7) we have

$$\frac{\int_{z_1}^{z_3} \sqrt{Q(z)}\,dz}{\int_{z_1}^{z_2} \sqrt{Q(z)}\,dz} = \frac{K + L e^{-i\gamma}}{M + N e^{-i\gamma}} \tag{13}$$

where $K=A_1 C_2-A_2 C_1$, $L=A_1 C_0-A_0 C_1$, $M=A_1 B_2-A_2 B_1$, $N=A_1 B_0-A_0 B_1$, and A_k, B_k and C_k are defined by

$$A_k=\int_{|z|=1} z^k D(z)\, dz\,, \quad B_k=\int_{z_1}^{z_2} z^k D(z)\, dz\,, \quad C_k=\int_{z_1}^{z_3} z^k D(z)\, dz\,,$$

for $k=0, 1, 2$. Here $1/D^2(z)$ denotes $(z-z_1)(1-\overline{z}_1 z) \ldots (z-z_4)(1-\overline{z}_4 z)$. Ultimately, we state our result as

Theorem. The variability region E of the cross ratio $R(F)$ of four points over the class Σ is a closed and connected set whose boundary is the image of an arc of the circumference $c(\gamma)=(M e^{i\gamma}+N)/(K e^{i\gamma}+L)$ under the elliptic modular function.

We conclude our paper with the following remarks :

Remark 1. The above theorem contains several well-known results cocerning variability regions of such quantities as $f(z)$, $zf'(z)/f(z)$ and $f(z_1)/f(z_2)$ in the class S and related classes. Unfortunately, the form of our solution does not provide a simple way of showing these results.

Remark 2. One can carry out considerations concerning elliptic and hyperelliptic integrals that occur in (10) along the lines presented in [2]. This will give the form of the extremal functions and the formula (12).

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ÖZET

F fonksiyonu Σ sınıfından olmak üzere $|z|>1$ bölgesinde verilen dört farklı noktanın F altındaki resimlerinin çifte oranı $R(F)$ olsun. Bu makalede, $R(F)$ çifte oranlarının Σ üzerinde değişim bölgesi eliptik modüler fonksiyon cinsinden bulunmaktadır.