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APPLICATION OF AVERAGING FUNCTIONAL CORRECTIONS METHOD TO NONLINEAR SINGULAR INTEGRAL EQUATIONS

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The sufficient conditions for the convergence of the averaging functional corrections method for the solution of a class of nonlinear singular integral equations with Hilbert kernel are discussed.

INTRODUCTION

Nonlinear singular integral equations with Cauchy and Hilbert kernel have been considered for long time. We refer to some well-known works of Guseinov A.I. and Mukhtarove Kh. Sh. [1], Musaev B.I. [1], Pogorzelski W. [6] and Wolfersdorf L.V. [1].

In the present paper the following nonlinear singular integral equation :

$$\varphi(t) = \lambda G\left(t, \frac{1}{2\pi}\int_0^{2\pi} g[t, \sigma, \varphi(\sigma)]\cot\frac{\sigma - t}{2}\, d\sigma\right) + f(t) \tag{1}$$

is investigated by means of the method of averaging functional corrections [1, 4, 5] in the real space $L_2[0, 2\pi]$, where $f(t) \in L_2[0, 2\pi]$ and λ is numerical parameter.

The idea of this method as applied to equation (1) is that, we take arbitrary element $\varphi_0(t) \in L_2[0, 2\pi]$, as an initial approximation and the next approximations are calculated from the relation :

$$\varphi_n(t) = \lambda G\left(t, \frac{1}{2\pi}\int_0^{2\pi} g[t, \sigma, \varphi_{n-1}(\sigma)]\cot\frac{\sigma - t}{2}\, d\sigma\right) + \lambda\alpha_n(t) + f(t) \tag{2}$$

where

$\alpha_n(t) = \sum_{i=1}^{r} c_{n_i}\Psi_i(t)$ and $\{\Psi_i(t)\}$ is an orthonormal system of linearly independent functions belonging to $L_2[0, 2\pi]$. The coefficients c_{n_i} are determined from the system :

$$c_{n_i} = \int_0^{2\pi} \delta_n(t)\ \Psi_i(t)\ dt,\ i = 1, 2, \ldots, r \tag{3}$$

where

$$\delta_n(t) = G\left(t, \frac{1}{2\pi} \int_0^{2\pi} g[t, \sigma, \varphi_n] \cot \frac{\sigma - t}{2}\, d\sigma\right) - \\ - G\left(t, \frac{1}{2\pi} \int_0^{2\pi} g[t, \sigma, \varphi_{n-1}(\sigma)] \cot \frac{\sigma - t}{2}\, d\sigma\right). \tag{4}$$

We shall prove the convergence of the iteration process (2), the existence of the solution of equation (1) and the solvability of system (3).

Let the function $G(t, u)$ be defined for $0 \leq t \leq 2\pi, -\infty < u < \infty$ and satisfy the conditon :

$$|G(t, u_1) - G(t, u_2)| \leq M |u_1 - u_2|, \tag{5}$$
$$G(t, 0) \in L_2 [0, 2\pi],$$

but the function $g[t, \sigma, \varphi(\sigma)]$ be defined for $0 \leq t, \sigma \leq 2\pi, -\infty < \varphi < \infty$ and satisfy the conditions :

$$|g[\sigma, \sigma, \varphi(\sigma)] - g[\sigma, \sigma, w(\sigma)]| \leq A |\varphi - w|, \tag{6}$$

$$|F[t, \sigma, \varphi(\sigma)] - F[t, \sigma, w(\sigma)]| \leq K(t, \sigma) |\varphi - w|, \tag{7}$$

where

$F[t, \sigma, \varphi(\sigma)] = g[t, \sigma, \varphi(\sigma)] - g[\sigma, \sigma, \varphi(\sigma)]$ and, $K(t, \sigma)$ is a nonnegative function such that :

$$\int_0^{2\pi} \int_0^{2\pi} K^2(t, \sigma) \left|\cot \frac{\sigma - t}{2}\right|^2 d\sigma\, dt = \rho_1^2 < \infty ; \tag{8}$$

M, A, ρ_1 are constants.

Let $\Delta_n(t) = \varphi^*(t) - \varphi_n(t)$, where $\varphi^*(t)$ and $\varphi_n(t)$ are the exact and the approximate solutions of equation (1) respectively.

From (1) and (2) we obtain :

$$|\Delta_n(t)| \leq \frac{|\lambda| M}{2\pi} \int_0^{2\pi} |g[t, \sigma, \varphi^*(\sigma)] - g[t, \sigma, \varphi_n(\sigma)]| \left|\cot \frac{\sigma - t}{2}\right| d\sigma + \\ + |\lambda|\ |\delta_n(t) - \alpha_n(t)|$$

using conditions (6), (7) and (8) and applying Hölder's inequality and Riesz's theorem [1, 3]. Finally, we obtain

$\| \Delta_n(t) \| \leq |\lambda| M(\rho_1 + AB) \| \Delta(t) \| + |\lambda| \| \delta_n(t) - \alpha_n(t) \|$, from which

$$\| \Delta_n(t) \| \leq \frac{|\lambda|}{1 - |\lambda| \rho_2} \| \delta_n(t) - \alpha_n(t) \| , \tag{9}$$

where $\rho_2 = M(\rho_1 + AB)$ and B is the norm of linear singular integral operator in $L_2[0, 2\pi]$. From (4), we have

$$\| \delta_n(t) \| \leq \rho_2 \| \varphi_n - \varphi_{n-1} \| , \tag{10}$$

since,

$$\varphi_n(t) - \varphi_{n-1}(t) = \lambda [\delta_{n-1}(t) - \alpha_{n-1}(t) + \alpha_n(t)],$$

then,

$$\| \delta_n(t) \| \leq |\lambda| \rho_2 \| \delta_{n-1}(t) - \alpha_{n-1}(t) + \alpha_n(t) \| , \tag{11}$$

it is easy to see that :

$$\| \delta_n(t) - \alpha_n(t) + \alpha_n(t) \|^2 = \| \delta_n(t) - \alpha_n(t) \|^2 + \| \alpha_n(t) \|^2. \tag{12}$$

From (10) and (11) we have

$$\| \delta_n(t) - \alpha_n(t) \|^2 + \| \alpha_n(t) \|^2 \leq (|\lambda| \rho_2)^2 [\| \delta_{n-1}(t) - \alpha_{n-1}(t) \|^2 + \| \alpha_n(t) \|^2].$$

If we choose λ such that :

$$|\lambda| \rho_2 < 1, \tag{13}$$

then,

$$\| \delta_n(t) - \alpha_n(t) \| \leq (|\lambda| \rho_2)^{n-1} \| \delta_1(t) - \alpha_1(t) \|. \tag{14}$$

From (9) and (14) we obtain

$$\| \Delta_n(t) \| \leq \frac{|\lambda| (|\lambda| \rho_2)^{n-1}}{1 - |\lambda| \rho_2} \| \delta_1(t) - \alpha_1(t) \|. \tag{15}$$

From here follows the convergence in norm in $L_2[0, 2\pi]$ of the successive approximations (2) to the solution of equation (1). To prove the existence of the solution of equation (1) we rewrite this equation in the operator form :

$$\varphi = S\varphi, \tag{16}$$

where

$$S\varphi = \lambda G\left(t, \frac{1}{2\pi} \int_0^{2\pi} g[t, \sigma, \varphi(\sigma)] \cot \frac{\sigma - t}{2} \, d\sigma\right) + f(t).$$

Let conditions (5) - (8) be satisfied and apply Hölder's integral inequality and Riez's theorem, then, finally we obtain

$$\| S\varphi_1 - S\varphi_2 \| \leq (|\lambda| p_2) \| \varphi_1 - \varphi_2 \|.$$

Thus by condition (13) it follows that the operator S is contraction mapping in $L_2[0, 2\pi]$, from which it follows the existence and uniqueness of the solution of equation (1) in $L_2[0, 2\pi]$.

Now, we show the solvability of the system (3). Substituting from (4) in (3) we have

$$c_{n_i} = \int_0^{2\pi} \left[G\left(t, \frac{1}{2\pi} \int_0^{2\pi} g[t, \sigma, \theta(\sigma)] \cot \frac{\sigma - t}{2} \, d\sigma\right) - \right.$$

$$\left. - G\left(t, \frac{1}{2\pi} \int_0^{2\pi} g[t, \sigma, \varphi_{n-1}(\sigma)] \cot \frac{\sigma - t}{2} \, d\sigma\right)\right] \Psi_i(t) \, dt,$$

where

$$\theta(\sigma) = \lambda G\left(\sigma, \frac{1}{2\pi} \int_0^{2\pi} g[\sigma, \tau, \varphi_{n-1}(\tau)] \cot \frac{\tau - \sigma}{2} \, d\tau\right) +$$

$$+ \lambda \sum_{j=1}^{r} c_{n_j} \Psi_i(\sigma) + f(\sigma). \tag{17}$$

Thus

$$\alpha_n(t) = \sum_{i=1}^{r} c_{n_i} \Psi_i(t) = \int_0^{2\pi} H_r(t, x) \left[G\left(t, \frac{1}{2\pi} \int_0^{2\pi} g[x, \sigma, \theta(\sigma)] \cot \frac{\sigma - x}{2} \, d\sigma\right) - \right.$$

$$\left. - G\left(x, \frac{1}{2\pi} \int_0^{2\pi} g[x, \sigma, \varphi_{n-1}(\sigma)] \cot \frac{\sigma - x}{2} \, d\sigma\right)\right] dx, \tag{18}$$

where

$$H_r(t, x) = \sum_{i=1}^{r} \Psi_i(t) \Psi_i(x).$$

We write (18) in the operator form :

$$\alpha_n = P_k E \alpha_n \tag{19}$$

where P_k is an orthogonal projection operator of the space $L_2[0, 2\pi]$ onto its subspace and $\| P_k \| = 1$, but

$$E\alpha_n = G\left(t, \frac{1}{2\pi}\int_0^{2\pi} g[t, \sigma, \theta(\sigma)] \cot\frac{\sigma - t}{2}\, d\sigma\right) -$$

$$- G\left(t, \frac{1}{2\pi}\int_0^{2\pi} g[t, \sigma, \varphi_{n-1}(\sigma)] \cot\frac{\sigma - t}{2}\, d\sigma\right).$$

Similar to evaluation of $\| S\varphi_1 - S\varphi_2 \|$, we obtain :

$$\| P_k E\alpha_n - P_k E\tilde{\alpha}_n \| \leq (|\lambda| \rho_2) \| \alpha_n - \tilde{\alpha}_n \|,$$

from here if condition (13) is satisfied, it follows that $P_k E$ is contraction mapping, consequently the corrections $\alpha_n(t)$ can be chosen uniquely.

To determine the successive approximations to the solution of equation (1) we can still consider such an algorithm of the method of averaging functional corrections :

$$\bar{\varphi}_0(t) \in L_2[0, 2\pi],$$

$$\bar{\varphi}_n(t) = \lambda G\left(t, \frac{1}{2\pi}\int_0^{2\pi} g[t, \sigma, \bar{\varphi}_{n-1}(\sigma) + \bar{\alpha}_n(\sigma)] \cot\frac{\sigma - t}{2}\, d\sigma\right) + f(t), \quad (20)$$

where

$\bar{\alpha}_n(t) = \sum_{=1}^{r} \bar{c}_{ni} \Psi_i(t)$ and the coefficients $\bar{c}_{ni}$ are defined from the set of equations :

$$\bar{c}_{ni} = \lambda \int_0^{2\pi} \left[G\left(t, \frac{1}{2\pi}\int_0^{2\pi} g\left[t, \sigma, \bar{\varphi}_{n-1}(\sigma) + \sum_{i=1}^{r} \bar{c}_{ni} \Psi_i(\sigma)\right] \cot\frac{\sigma - t}{2}\, d\sigma\right) - \right.$$

$$\left. - G\left(t, \frac{1}{2\pi}\int_0^{2\pi} g[t, \sigma, \bar{\varphi}_{n-2}(\sigma) + \bar{\alpha}_{n-1}(\sigma)] \cot\frac{\sigma - t}{2}\, d\sigma\right)\right] \Psi_i(t)\, dt. \quad (21)$$

The sequence $\{\bar{\varphi}_n(t)\}$ converges in norm in $L_2[0, 2\pi]$ to the solution of equation (1) if $|\lambda| \rho_2 < 1$, that is as above we have

$$\| \bar{\Delta}_n(t) \| \leq \frac{|\lambda| (|\lambda| \rho_2)^{n-1}}{i - |\lambda| \rho_2} \| \bar{\delta}_1(t) - \bar{\alpha}_1(t) \|. \quad (22)$$

Thus the following theorem is proved :

Theorem. Let the function $G(t, u)$ be defined at $0 \leq t \leq 2\pi$, $-\infty < u < \infty$ and satisfy the condition (5), but the function $g[t, \sigma, \varphi]$ be defined at $0 \leq t, \sigma \leq 2\pi$, $-\infty < \varphi < \infty$ and satisfy the conditions (6) - (8), λ satisfy the condition (13).

Then equation (1) has unique solution in $L_2[0, 2\pi]$ to which the successive approximation (2) or (20) converges, moroever the inequality (15) or (22) is valid.

The algorithm (2) converges under the same conditions as the method of successive approximations, but the introducing of functional corrections accelerates the convergence.

Consider the two special cases of equation (1), that is the equations of the form :

$$\varphi(t) = \frac{\lambda}{2\pi} \int_0^{2\pi} g[t, \sigma, \varphi(\sigma)] \cot \frac{\sigma - t}{2} d\sigma + f(t) \tag{23}$$

and

$$\varphi(t) = \frac{\lambda}{2\pi} \int_0^{2\pi} g[\sigma, \varphi(\sigma)] \cot \frac{\sigma - t}{2} d\sigma + f(t) \tag{24}$$

in the real space $L_2[0, 2\pi]$. Take the initial approximation and an arbitrary element $\varphi_0(t) \in L_2[0, 2\pi]$, but the next approximations defined by the relations:

$$\varphi_n(t) = \frac{\lambda}{2\pi} \int_0^{2\pi} g[t, \sigma, \varphi_{n-1}(\sigma)] \cot \frac{\sigma - t}{2} d\sigma + \lambda \alpha_n + f(t), \tag{25}$$

and

$$\varphi_n(t) = \frac{\lambda}{2\pi} \int_0^{2\pi} g[\sigma, \varphi_{n-1}(\sigma)] \cot \frac{\sigma - t}{2} d\sigma + \varphi \alpha_n + f(t), \tag{26}$$

respectively.

Corollary 1. Let the function $g[t, \sigma, \varphi_{n-1}(\sigma)]$ be defined at $0 \leq t, \sigma \leq 2\pi$, $-\infty < \varphi < \infty$ and satisfy the conditions (σ) and (7), and the parameter λ satisfy the condition $|\lambda|(\rho_1 + AB) < 1$. Then the equation (23) has unique solution in $L_2[0, 2\pi]$ to which the successive approximations (25) converge, moreover the following inequality is valid :

$$\| \Delta_n(t) \| \leq \frac{|\lambda| \eta^{n-1}}{1 - \eta} \| \delta_1(t) - \alpha_1(t) \|,$$

where

$$\eta = |\lambda|(\rho_1 + AB).$$

Corollary 2. Let the function $g[\sigma, \varphi(\sigma)]$ be defined at $0 \leq \sigma \leq 2\pi$, $-\infty < \varphi < \infty$ and satisfy the condition

$$|g[\sigma, \varphi_1] - g[\sigma, \varphi_2]| \leq M |\varphi_1 - \varphi_2|,$$
$$g[\sigma, 0] \in L_2[0, 2\pi]$$

and the parameter λ satisfy the condition $|\lambda| MB < 1$. Then the equation (24) has unique solution in $L_2[0, 2\pi]$ to which successive approximation (26) converges, moreover the following inequality is valid :

$$\| \Delta_n(t) \| \leq \frac{|\lambda| \zeta^{n-1}}{1-\zeta} \| \delta_1(t) - \alpha_1(t) \|,$$

where

$$\zeta = |\lambda| MB < 1.$$

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ÖZET

Bu çalışmada, Hilbert çekirdekli ve lineer olmayan bir tür singüler integral denklemin çözümü ile ilgili fonksiyonel düzeltmeler metodunun yakınsaklığı için yeter koşullar verilmektedir.