SURFACES ON WHICH THE UNION CURVES FORM AN HEXAGONAL THREE - WEB *)

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The congruence formed by the intersection of the osculating planes of the two lines of curvature of a surface S, cutting each other in each point, has been considered and the conditions for any three families of union curves relative to this congruence to form an hexagonal three - web have established. Exact results have been obtained for those surfaces whose Gaussian Curvature, when S is referred to its lines of curvature, takes the form

$$K = U_1(u^1) \, . \, U_2(u^2) \neq 0.$$

0. Introduction. The problem of studying the families of curves drawn on a surface and forming an hexagonal web has been considered by various authours. The first result on this subject was given by THÖMSEN [1]. According to this result, the Darboux Lines form an hexagonal web if and only if they are on an Isothermic Asymptotic Surface [2]. The problem considered in this thesis is that of determining those surfaces on which the union curves form an hexagonal three - web.

A union curve on a surface, relative to a given congruence, has the property that its osculating plane at each point of the curve, contains the line of the congruence through the point. In general, the differential equation of the union curves is an hypergeodesic equation and such a differential equation can be integrated only in very few cases. In this thesis, a congruence, which depends on the geometrical properties of the surface, has been chosen. In doing so, care has been taken that only two families of union curves relative to this congruence coincide with the two families of parametric lines of the

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surface. Because of this specialisation, it is possible to integrate the differential equation of union curves. All the surfaces taken into consideration are referred to their lines of curvature and for non-developable surfaces, the congruence formed by the intersection of the osculating planes of the two lines of curvature cutting each other at that particular point of the surface has been chosen. It is seen that, the differential equation of union curves with respect to this congruence, is a special hypergeodesic equation and can be integrated easily for a large class of surfaces, in particular for those surfaces whose Gaussian curvature is of the form

$$K = U_1(u^1) \cdot U_2(u^2) \neq 0$$

when referred to their lines of curvature.

The thesis is divided into two chapters. Chapter one contains the various preliminary definitions required for subsequent developments, such as those of union curves and hexagonal three - webs. In chapter two, a necessary and sufficient condition is found for the surfaces whose Gaussian curvature is of the form $K = U_1(u^1) \cdot U_2(u^2) \neq 0$ when referred to their lines of curvature, for the two families of lines of curvature together with a family of union curves to form an hexagonal three - web. This condition remains the same for any three families of union curves, relative to the congruence which is mentioned above, to form an hexagonal three - web. Besides, in the same chapter a further sufficient condition is given for any three families of union curves, which are on a non-developable surface referred to its lines of curvature to form an hexagonal three-web. Again for non-developable isothermic surfaces another sufficient condition is given. As a consequence of all these, it is shown that on quadric surfaces, surfaces of revolution, isothermic surfaces which are applicable to a surface of revolution, Duplin's Cyclides and surfaces parallel to Dupin's Cyclides, to surfaces of revolution and to isothermic surfaces which are applicable to a surface of revolution, the union curves form hexagonal three - web and a necessary and sufficient condition for the union curves of a pseudo - spherical surface to form an hexagonal three - web also.

CHAPTER I

I. 1. Union Curves. A union curve on a surface S, relative to a given congruence $\mathscr{L}$, has the property that its osculating plane at each point P of the curve C contains the line of the congruence through this point [3].

Let the surface S be defined analytically with reference to an orthogonal cartesian system of coordinates by [1)]

$$x^i = x^i(u^1, u^2) \quad , \quad (i = 1,2,3) \tag{1.1}$$

where the functions x^i and their partial derivatives to the second order are understood to be continuous at any point P on S. Let the line l of the congruence $\mathscr{L}$ at P have direction cosines given by

$$\lambda^i = \lambda^i(u^1, u^2) \quad , \quad \delta_{ij}\, \lambda^i \lambda^j = 1$$

where the functions λ^i are continuous at P. If X^i denote the direction cosines of the normal to S at P, the direction cosines of l at P may be written in the form

$$\lambda^i = p^\alpha x^i{}_{,\alpha} + qX^i \quad , \quad (q > 0) \tag{1.2}$$

where, for convenience, the notation of the covariant derivative $x^i{}_{,\alpha}$ of x^i with respect to the first fundamental tensor $(g_{\alpha\beta} = x^i{}_{,\alpha}\, x^i{}_{,\beta})$ of S is used instead of $\dfrac{\partial x^i}{\partial u^\alpha}$. Making necessary calculations, one finds the following differential equation of union curves on S [4]

$$\varepsilon_{\sigma\tau}\, u^{\sigma\prime}\, (q\,\rho^\tau - \kappa p^\tau) = 0 \tag{1.3}$$

where $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$ and $\rho^\tau = u^{\tau\prime\prime} + \Gamma^\tau_{\alpha\beta}\, u^{\alpha\prime}\, u^{\beta\prime}$ are the components of the curvature vector of C at P and $\kappa = d_{\alpha\beta}\, u^{\alpha\prime}\, u^{\beta\prime}$ is the normal curvature of the curve C with direction $u^{\alpha\prime}$ on the surface S. Equation (1.3) is the differential equation of second order of the union curves on the surface S through any point P on S, the parametric lines being arbitrary. Expanding (1.3) one finds [5, 215]

$$\begin{aligned} \frac{d^2 u^2}{du^1\, du^1} &+ (h^1 d_{22} - \Gamma^1_{22}) \left(\frac{du^2}{du^1}\right)^3 + (\Gamma^2_{22} - 2\Gamma^1_{12} + 2h^1 d_{12} \\ &- h^2 d_{22}) \left(\frac{du^2}{du^1}\right)^2 + (2\Gamma^2_{12} - \Gamma^1_{11} + h^1 d_{11} \\ &- 2h^2 d_{12}) \left(\frac{du^2}{du^1}\right) + (\Gamma^2_{11} - h^2 d_{11}) = 0 \end{aligned} \tag{1.4}$$

1) Greek indices will always take the range 1,2 and Latin indices the range 1,2,3.

where $h^\alpha = \frac{p^\alpha}{q}$ · From the equation (1.4) it is seen that the coordinate curves $du^1 = 0$, $du^2 = 0$ on the surface S are union curves if and only if

$$h^\alpha d_{\beta\beta} - \Gamma^\alpha_{\beta\beta} = 0 \quad , \quad (\alpha, \beta = 1,2 \quad ; \quad \alpha \neq \beta).$$

I. 2. Conditions for the curves on a surface to form an hexagonal three - web. [6] Let the curves C_j be defined by the equations

$$C_j \quad : \quad du^2 + f_j(u^1, u^2)\, du^1 = 0 \qquad (j = 1,2,3)$$

on the surface S. It is easily seen that, these three families of curves form an hexagonal three - web [7, 164] if

$$(1.5) \quad \frac{\partial}{\partial u^2} \left\{ \frac{r_{11}(r_{22,1} - r_{21,2}) - r_{21}(r_{12,1} - r_{11,2})}{r_{11}\, r_{22} - r_{21}\, r_{12}} \right\} = \frac{\partial}{\partial u^1} \left\{ \frac{r_{12}(r_{22,1} - r_{21,2}) - r_{22}(r_{12,1} - r_{11,2})}{r_{11}\, r_{22} - r_{21}\, r_{12}} \right\}$$

holds, where

$$r_{j1}(u^1, u^2) = f_k(u^1, u^2) - f_t(u^1, u^2)$$

$$r_{j2}(u^1, u^2) = f_j(u^1, u^2)\, \{f_k(u^1, u^2) - f_t(u^1, u^2)\}$$

$$(j \neq k \neq t \neq j = 1,2,3).$$

In particular, if the curves are of the form

$$du^2 = 0 \quad , \quad du^1 = 0 \quad , \quad du^2 - f(u^1, u^2)\, du^1 = 0$$

the condition (1.5) reduces to

$$(1.6) \qquad \frac{\partial^2(\log f)}{\partial u^1\, \partial u^2} = (\log f)_{,12} = 0$$

and therefore the function $f(u^1, u^2)$ is of the form

$$(1.7) \qquad f(u^1, u^2) = U^1(u^1) \cdot U^2(u^2).$$

CHAPTER II

II. 1. Union curves on non - developable surfaces. On a regular surface S, we can always choose the lines of curvature as parametric lines. For such a choice on a non - developable surface ($g_{12} = d_{12} = 0$, $d_{11} \neq 0$, $d_{22} \neq 0$) the differential equ ation of union curves relative to the congruence

$$\mathscr{L} \ : \ z^i = x^i + t(h^\alpha x^i{}_{,\alpha} + X^i) \tag{2.1}$$

is

$$\frac{d^2 u^2}{du^1\, du^1} + (h^1 d_{22} - \Gamma^1_{22}) \left(\frac{du^2}{du^1}\right)^3 + (\Gamma^2_{22} - 2\Gamma^1_{12} - h^2 d_{22}) \left(\frac{du^2}{du^1}\right)^2 + (2\Gamma^2_{12} - \Gamma^1_{11} + h^1 d_{11}) \left(\frac{du^2}{du^1}\right) + (\Gamma^2_{11} - h^2 d_{22}) = 0. \tag{2.2}$$

Equation (2.2) is an hypergeodesic equation of the form

$$\frac{d^2 u^2}{du^1\, du^1} = A_1(u^1, u^2) \left(\frac{du^2}{du^1}\right)^3 + A_2(u^1, u^2) \left(\frac{du^2}{du^1}\right)^2 + A_3(u^1, u^2) \left(\frac{du^2}{du^1}\right) + A_4(u^1, u^2)$$

and such a differential equation can be integrated only in very few special cases [[3], 217].

Let the congruence formed by the intersection of the osculating planes of the two lines of curvature cutting each other at that particular point of the surface he chosen. In this case, the coordinate curves on the surface are also union curves and equation (2.2) has the form

$$\frac{d^2 u^2}{du^1\, du^1} + \left(\Gamma^2_{22} - 2\Gamma^1_{12} - \frac{d_{22}}{d_{11}}\, \Gamma^2_{11}\right) \left(\frac{du^2}{du^1}\right)^2 + \left(2\Gamma^2_{12} - \Gamma^1_{11} + \frac{d_{11}}{d_{22}}\, \Gamma^1_{22}\right) \left(\frac{du^2}{du^1}\right) = 0, \tag{2.3}$$

where

$$\mathscr{L}^* \ : \ z^{*i} = x^i + t\left(\frac{\Gamma^\alpha_{\beta\beta}}{d_{\beta\beta}}\, x^i{}_{,\alpha} + X^i\right) \quad , \quad (\alpha \neq \beta). \tag{2.4}$$

If we write

$$A_\alpha(u^1, u^2) = \Gamma^\beta_{\beta\beta} - 2\Gamma^\alpha_{\alpha\beta} - \frac{d_{\beta\beta}}{d_{\alpha\alpha}} \Gamma^\beta_{\alpha\alpha} \quad , \quad (\alpha \neq \beta)$$

and

$$\varepsilon_{11} = \varepsilon_{22} = 0 \quad , \quad \varepsilon_{12} = -\varepsilon_{21} = 1$$

equation (2.3) takes the form

$$\text{(2.5)} \qquad \frac{d^2 u^2}{du^1\, du^1} + \varepsilon_{\alpha\beta} A_\alpha(u^1, u^2) \left(\frac{du^2}{du^1}\right)^\beta = 0 \quad , \quad (\alpha \neq \beta).$$

On the other hand, since $d_{\alpha\beta} = 0$ for the surface S, the Codazzi equations

$$\frac{\partial d_{\alpha\alpha}}{\partial u^\beta} - \frac{\partial d_{\alpha\beta}}{\partial u^\alpha} = d_{\alpha\gamma}\, \Gamma^\gamma_{\alpha\beta} - d_{\gamma\beta}\, \Gamma^\gamma_{\alpha\alpha} \quad , \quad (\alpha \neq \beta)$$

take the form

$$\frac{d_{\beta\beta}}{d_{\alpha\alpha}} \Gamma^\beta_{\alpha\alpha} = \Gamma^\alpha_{\alpha\beta} - \frac{\partial(\log\, d_{\alpha\alpha})}{\partial u^\beta} \quad , \quad (\alpha \neq \beta)$$

and therefore we have

$$\begin{aligned} A_\alpha(u^1, u^2) &= \Gamma^\beta_{\beta\beta} - 3\Gamma^\alpha_{\alpha\beta} + \frac{\partial(\log\, d_{\alpha\alpha})}{\partial u^\beta} \\ &= (\log \sqrt{g_{\beta\beta}})_{,\beta} - 3(\log \sqrt{g_{\alpha\alpha}})_{,\beta} + (\log d_{\alpha\alpha})_{,\beta} \\ &= \left(\log \frac{d_{\alpha\alpha} \sqrt{g_{\beta\beta}}}{g_{\alpha\alpha} \sqrt{g_{\alpha\alpha}}}\right)_{,\beta} \qquad (\alpha \neq \beta). \end{aligned}$$

Setting the above values for A_α in (2.5) we find that

$$\frac{d^2 u^2}{du^1\, du^1} + \varepsilon_{\alpha\beta} \left(\log \frac{d_{\alpha\alpha} \sqrt{g_{\beta\beta}}}{g_{\alpha\alpha} \sqrt{g_{\alpha\alpha}}}\right)_{,\beta} \cdot \left(\frac{du^2}{du^1}\right)^\beta = 0 \quad (\alpha, \beta = 1,2 \; ; \; \alpha \neq \beta)$$

and hence

$$\frac{du^2}{du^1}\left\{\frac{\dfrac{d^2 u^2}{du^1\, du^1}}{\dfrac{du^2}{du^1}} + \varepsilon_{\alpha\beta}\left(\log \frac{d_{\alpha\alpha}\sqrt{g_{\beta\beta}}}{g_{\alpha\alpha}\sqrt{g_{\alpha\alpha}}}\right)_{,\beta} \cdot \left(\frac{du^2}{du^1}\right)^{\beta-1}\right\} = 0$$

thus

$$\frac{du^2}{du^1} = 0$$

and

$$\frac{d\left(\log \dfrac{du^2}{du^1}\right)}{du^1} + \varepsilon_{\alpha\beta}\left(\log \frac{d_{\alpha\alpha}\sqrt{g_{\beta\beta}}}{g_{\alpha\alpha}\sqrt{g_{\alpha\alpha}}}\right)_{,\beta} \cdot \left(\frac{du^2}{du^1}\right)^{\beta-1} = 0$$

or, since $\beta = 1,2$,

$$d\left(\log \frac{du^2}{du^1}\right) + \varepsilon_{\alpha\beta}\left(\log \frac{d_{\alpha\alpha}\sqrt{g_{\beta\beta}}}{g_{\alpha\alpha}\sqrt{g_{\alpha\alpha}}}\right)_{,\beta} du^\beta = 0 \tag{2.6}$$

are the differential equations of the union curves on the surface S through any point P on S.

If the integrability condition

$$\left(\log \frac{d_{\alpha\alpha}\sqrt{g_{\beta\beta}}}{g_{\alpha\alpha}\sqrt{g_{\alpha\alpha}}}\right)_{,\beta\alpha} = -\left(\log \frac{d_{\beta\beta}\sqrt{g_{\alpha\alpha}}}{g_{\beta\beta}\sqrt{g_{\beta\beta}}}\right)_{,\alpha\beta} \tag{2.7}$$

of the differential equation (2.6) is satisfied, we can write

$$\varepsilon_{\alpha\beta}\left(\log \frac{d_{\alpha\alpha}\sqrt{g_{\beta\beta}}}{g_{\alpha\alpha}\sqrt{g_{\alpha\alpha}}}\right)_{,\beta} du^\beta = d\psi(u^1,\, u^2)$$

and the equation (2.6) takes the form

$$d\left(\log \frac{du^2}{du^1}\right) + d\psi(u^1,\, u^2) = 0\,, \tag{2.8}$$

which can be integrated easily.

Since the surface S referred to its lines of curvature, the Gaussian curvature K of S is of the form

$$K = \frac{d_{\alpha\alpha} \cdot d_{\beta\beta}}{g_{\alpha\alpha} \cdot g_{\beta\beta}} \quad , \quad (\alpha \neq \beta).$$

On the other hand, the condition (2.7) is

$$\left(\log \frac{d_{\alpha\alpha} \sqrt{g_{\beta\beta}}}{g_{\alpha\alpha} \sqrt{g_{\alpha\alpha}}} + \log \frac{d_{\beta\beta} \sqrt{g_{\alpha\alpha}}}{g_{\beta\beta} \sqrt{g_{\beta\beta}}}\right)_{,\alpha\beta} = \left(\log \frac{d_{\alpha\alpha} \cdot d_{\beta\beta}}{g_{\alpha\alpha} \cdot g_{\beta\beta}}\right)_{,\alpha\beta} = 0 \,, \quad (\alpha \neq \beta).$$

So, for the surfaces S, which satisfy the condition

$$\left(\log \frac{d_{\alpha\alpha} \cdot d_{\beta\beta}}{g_{\alpha\alpha} \cdot g_{\beta\beta}}\right)_{,\alpha\beta} = (\log\ K)_{,\alpha\beta} = 0$$

that is to say for those surfaces for which the Gaussian curvature is of the form

$$K = U_1(u^1) \cdot U_2(u^2), \tag{2.9}$$

the differential equation (2.6) of the union curves on S reduces to (2.8). In this case, since

$$\frac{\partial \psi}{\partial u^\alpha} = \psi_{,\alpha} = \varepsilon_{\alpha\beta} \left(\log \frac{g_{\beta\beta} \sqrt{g_{\beta\beta}}}{d_{\beta\beta} \sqrt{g_{\alpha\alpha}}}\right)_{,\alpha} \quad , \quad (\alpha, \beta = 1,2 \ ; \ \alpha \neq \beta) \tag{2.10}$$

then

$$\psi(u^1,\ u^2) = \varepsilon_{\alpha\beta} \left(\log \frac{g_{\beta\beta} \sqrt{g_{\beta\beta}}}{d_{\beta\beta} \sqrt{g_{\alpha\alpha}}}\right) + \eta(u^\beta)$$

and

$$\psi_{,\beta} = \varepsilon_{\alpha\beta} \left(\log \frac{g_{\beta\beta} \sqrt{g_{\beta\beta}}}{d_{\beta\beta} \sqrt{g_{\alpha\alpha}}}\right)_{,\beta} + \eta'(u^\beta) = \varepsilon_{\beta\alpha} \left(\log \frac{g_{\alpha\alpha} \sqrt{g_{\alpha\alpha}}}{d_{\alpha\alpha} \sqrt{g_{\beta\beta}}}\right)_{,\beta}$$

so, we find

$$\eta'(u^\beta) = \varepsilon_{\alpha\beta} \left(\log \frac{d_{\alpha\alpha} \cdot d_{\beta\beta}}{g_{\alpha\alpha} \cdot g_{\beta\beta}}\right)_{,\beta} .$$

On the other hand we have

$$\frac{d_{\alpha\alpha} \cdot d_{\beta\beta}}{g_{\alpha\alpha} \cdot g_{\beta\beta}} = K = U_1(u^1) \cdot U_2(u^2) \quad , \quad (\alpha \neq \beta)$$

then

$$\left(\log \frac{d_{\alpha\alpha} \cdot d_{\beta\beta}}{g_{\alpha\alpha} \cdot g_{\beta\beta}} \right)_{,\beta} = \frac{d(\log U_\beta)}{du^\beta}$$

and hence

$$\frac{d\eta(u^\beta)}{du^\beta} = \varepsilon_{\alpha\beta} \frac{d(\log U_\beta)}{du^\beta}$$

or

$$\eta(u^\beta) = \varepsilon_{\alpha\beta} \log U_\beta \,.$$

Therefore

$$\psi(u^1, u^2) = \varepsilon_{\alpha\beta} \left(\log \frac{g_{\beta\beta} \sqrt{g_{\beta\beta}}\, U_\beta}{d_{\beta\beta} \sqrt{g_{\alpha\alpha}}} \right) .$$

Finally the differential equation (2.8) reduces to

$$d \left(\log \frac{du^2}{du^1} \right) + \varepsilon_{\alpha\beta}\, d \left(\log \frac{g_{\beta\beta} \sqrt{g_{\beta\beta}}\, U_\beta}{d_{\beta\beta} \sqrt{g_{\alpha\alpha}}} \right) = 0$$

or

$$\varepsilon_{\alpha\beta}\, d \left(\log \frac{du^\beta}{du^\alpha} \right) + \varepsilon_{\alpha\beta}\, d \left(\log \frac{g_{\beta\beta} \sqrt{g_{\beta\beta}}\, U_\beta}{d_{\beta\beta} \sqrt{g_{\alpha\alpha}}} \right) = 0$$

and since $\varepsilon_{\alpha\beta} \neq 0$ we get

$$d \left(\log \frac{du^\beta}{du^\alpha} \right) + d \left(\log \frac{g_{\beta\beta} \sqrt{g_{\beta\beta}}}{d_{\beta\beta} \sqrt{g_{\alpha\alpha}}}\, U_\beta \right) = 0$$

so, by the first integration of this differential equation we obtain

$$du^\alpha - c \frac{g_{\beta\beta} \sqrt{g_{\beta\beta}}}{d_{\beta\beta} \sqrt{g_{\alpha\alpha}}}\, U_\beta\, du^\beta = 0 \quad , \quad (\alpha \neq \beta) \quad , \quad (c = \text{cons.}) . \tag{2.11}$$

II. 1.1. Conditions for the two families of lines of curvature together with a family of union curves to form an hexagonal three - web on a surface S.

Theorem II. 1. *On a surface S whose Gaussian curvature is of the form*

$$K = U_1(u^1) \cdot U_2(u^2)$$

when referred to its lines of curvature, the necessary and sufficient condition for the two families of lines of curvature together with a certain family of union curves relative to the congruence (2.4) *to form an hexagonal three - web is*

$$\left(\log \frac{g_{\beta\beta}\sqrt{g_{\beta\beta}}}{d_{\beta\beta}\sqrt{g_{\alpha\alpha}}}\right)_{,\alpha\beta} = 0 \quad \text{or} \quad \frac{g_{\alpha\alpha}\sqrt{g_{\alpha\alpha}}}{d_{\alpha\alpha}\sqrt{g_{\beta\beta}}} = f^1(u^1) \cdot f^2(u^2) \quad , \quad (\alpha \neq \beta).$$

Proof. For the surface in consideration, since the lines of curvature are parametric and the Gaussian curvature is of the form $K = U_1(u^1) \cdot U_2(u^2)$, the union curves relative to the congruence (2.4) are defined by the equation (2.11). Let us take the two families of lines of curvature

$$du^\alpha = 0 \qquad (\alpha = 1,2)$$

together with a family of union curves

$$du^\alpha - c\,\frac{g_{\beta\beta}\sqrt{g_{\beta\beta}}}{d_{\beta\beta}\sqrt{g_{\alpha\alpha}}}\,U_\beta\, du^\beta = 0 \qquad (\alpha \neq \beta)$$

which corresponds to a certain value of c. According to the equations (1.6) and (1.7) the necessary and sufficient condition for these three families of curves to form an hexagonal three - web is

$$\text{(2.12)} \quad \left(\log \frac{g_{\beta\beta}\sqrt{g_{\beta\beta}}}{d_{\beta\beta}\sqrt{g_{\alpha\alpha}}}\right)_{,\alpha\beta} = 0 \quad \text{or} \quad \frac{g_{\alpha\alpha}\sqrt{g_{\alpha\alpha}}}{d_{\alpha\alpha}\sqrt{g_{\beta\beta}}} = f^1(u^1) \cdot f^2(u^2) \quad , \quad (\alpha \neq \beta).$$

On the other hand, since the above condition does not contain specifically the elements of the family of union curves that has been chosen, one obtains the following theorem also :

Theorem II. 2. *If the two families of lines of curvature together with a certain family of union curves relative to the congruence (2.4) to form an hexagonal three - web on a surface S on which the lines of curvature are parametric and the Gaussian curvature is of the form* $K = U_1(u^1) \cdot U_2(u^2)$, *then the two families of lines of curvature and any one of the families of union curves relative to the same congruence will form an hexagonal three - web.*

II. 1.2. Conditions for any three families of union curves to form an hexagonal three - web on a surface S. Since the two families of lines of curvature mentioned in II. 1.1. are also union curves relative to the congruence (2.4), the equation (2.12) is the condition for three special families of union curves to form an hexagonal three - web on S. The following theorem gives a necessary and sufficient condition for any three families of union curves to form an hexagonal three - web on a surface S.

Theorem II. 3. *The necessary and sufficient condition for any three families of union curves relative to the congruence* (2.4) *to form an hexagonal three - web on a surface S on which the lines of curvature are parametric and the Gaussian curvature is of the form* $K = U_1(u^1) \cdot U_2(u^2)$ *is*

$$\left(\log \frac{g_{\alpha\alpha} \sqrt{g_{\alpha\alpha}}}{d_{\alpha\alpha} \sqrt{g_{\beta\beta}}}\right)_{,\alpha\beta} = 0 \quad \text{or} \quad \frac{g_{\alpha\alpha} \sqrt{g_{\alpha\alpha}}}{d_{\alpha\alpha} \sqrt{g_{\beta\beta}}} = f^1(u^1) \cdot f^2(u^2) \quad , \quad (\alpha \neq \beta).$$

Proof. Take any three families of union curves satisfying the equation (2.11) relative to the congruence (2.4), which correspond to the values c_j $(j = 1,2,3)$ of the constant c :

$$du^\alpha - c_j \frac{g_{\beta\beta} \sqrt{g_{\beta\beta}}}{d_{\beta\beta} \sqrt{g_{\alpha\alpha}}} U_\beta(u^\beta)\, du^\beta = 0 \quad , \quad (j = 1,2,3) \quad , \quad (\alpha \neq \beta)$$

if we put

$$\Phi(u^1, u^2) = \frac{g_{11} \sqrt{g_{11}}}{d_{11} \sqrt{g_{22}}} U_1(u^1) ,$$

from I. 2. we have

$$r_{j1} = (c_k - c_l)\ \Phi(u^1, u^2)$$

$$r_{j2} = c_j (c_k - c_l)\ [\Phi(u^1, u^2)]^2$$

and condition (1,5) takes to form

$$\frac{\partial}{\partial u^2}\left(2\,\frac{\Phi,_1}{\Phi}\right) = \frac{\partial}{\partial u^1}\left(2\,\frac{\Phi,_2}{\Phi}\right)$$

or

$$[\log\ \Phi(u^1,\ u^2)],_{12} = 0\,.$$

Thus

$$\left(\log\frac{g_{11}\sqrt{g_{11}}}{d_{11}\sqrt{g_{22}}}\right),_{12} = 0 \quad \text{or} \quad \left(\log\frac{d_{22}\sqrt{g_{11}}}{g_{22}\sqrt{g_{22}}}\right),_{12} = 0$$

are the necessary and sufficient condition for these three families of union curves to form an hexagonal three - web on S.

We now prove the following theorem :

Theorem II. 4. *If for a surface S which is referred to its lines of curvature, the coefficients g_{11} and g_{22} of the first fundamental form are both functions of u^1 alone or u^2 alone, then any three families of union curves relative to the congruence (2.4) form an hexagonal three - web on S.*

Proof. If $g_{11} = g_{11}(u^1)$ and $g_{22} = g_{22}(u^1)$ then from the following equation of Gauss

$$K = -\frac{1}{\sqrt{g_{\alpha\alpha}\cdot g_{\beta\beta}}}\left[\frac{\partial}{\partial u^\beta}\left(\frac{1}{\sqrt{g_{\beta\beta}}}\cdot\frac{\partial\sqrt{g_{\alpha\alpha}}}{\partial u^\beta}\right)\right]\ , \quad (\alpha,\beta = 1,2\ ;\ \alpha\neq\beta)$$

it is seen that $K = K(u^1)$. Thus, the hyphoteses of theorem II. 3. are satisfied. On the other hand, from the Codazzi equation

$$\frac{\partial d_{11}}{\partial u^2} = d_{11}\,\Gamma^1_{12} - d_{22}\,\Gamma^2_{11}$$

and from the relations

$$\Gamma^1_{12} = \Gamma^2_{11} = 0$$

we find that

$$\frac{\partial d_{11}}{\partial u^2} = 0$$

so

$$d_{11} = d_{11}(u^1)$$

and

$$\left(\log \frac{g_{11}\sqrt{g_{11}}}{d_{11}\sqrt{g_{22}}}\right)_{,12} = 0 .$$

The same is also valid for u^2 and hence the proof of the theorem is complete.

Application II. 1.1. Quadrics.

1. Central Quadrics. From the parametric equations of the central quadrics [8, 228]

$$x^j = \pm \sqrt{\frac{a_j(a_j - u^1)\,(a_j - u^2)}{(a_j - a_k)\,(a_j - a_t)}} \quad , \quad (j \neq k \neq t \neq j = 1,2,3)$$

we find

$$g_{\alpha\alpha} = \frac{u^\alpha(u^\alpha - u^\beta)}{f(u^\alpha)} \quad , \quad g_{\alpha\beta} = 0$$

and $(\alpha,\ \beta = 1,2\ ;\ \alpha \neq \beta)$

$$d_{\alpha\alpha} = -\sqrt{\frac{a_1\, a_2\, a_3}{u^1\, u^2}} \cdot \frac{a^\alpha - u^\beta}{f(u^\alpha)} \quad , \quad d_{\alpha\beta} = 0$$

where

$$f(x) = 4(a_1 - x)\,(a_2 - x)\,(a_3 - x) .$$

Since

$$K = \frac{d_{11} \cdot d_{22}}{g_{11} \cdot g_{22}} = \frac{a_1\, a_2\, a_3}{(a^1)^2 \cdot (u^2)^2} = U_1(u^1) \cdot U_2(u^2) \neq 0$$

and

$$\frac{g_{11}\sqrt{g_{11}}}{d_{11}\sqrt{g_{22}}} = -i\, \frac{(u^1)^2\sqrt{f(u^2)}}{\sqrt{a_1\, a_2\, a_3}\ \sqrt{f(u^1)}} = \alpha_1(u^1) \cdot \alpha_2(u^2) \quad , \quad (i^2 = -1)$$

these surfaces satisfy all the conditions of theorem II. 3.

2. Paraboloids. From the parametric equations of the paraboloids [9,131]

$$x^j = 2\sqrt{\frac{a_j(a_j - u^1)\,(a_j - u^2)}{(a_k - a_j)}} \ , \ (j \neq k = 1,2) \ , \ x^3 = u^1 + u^2 - a_1 - a_2$$

we find that

$$g_{\alpha\alpha} = \frac{u^\alpha(u^\alpha - u^\beta)}{f(u^\alpha)} \quad , \quad g_{\alpha\beta} = 0$$

and $(\alpha, \beta = 1,2 \; ; \; \alpha \neq \beta)$

$$d_{\alpha\alpha} = \sqrt{\frac{a_1 \, a_2}{u^1 \, u^2}} \cdot \frac{u^\alpha - u^\beta}{2 f(u^\alpha)} \quad , \quad d_{\alpha\beta} = 0$$

where

$$f(x) = (u_1 - x) \; (u_2 - x) .$$

Since

$$K = \frac{a_1 \, a_2}{4 (u^1)^2 \cdot (u^2)^2} = V_1(u^1) \cdot V_2(u^2) \neq 0$$

and

$$\frac{g_{11} \sqrt{g_{11}}}{d_{11} \sqrt{g_{22}}} = - i \, \frac{2(u^1)^2}{\sqrt{a_1 \, a_2}} \cdot \frac{\sqrt{f(u^2)}}{\sqrt{f(u^1)}} = \beta_1(u^1) \cdot \beta_2(u^2)$$

these surfaces also satisfy all the conditions of theorem II. 3. Thus the following theorem is obtained :

Theorem II. 5. *Any three families of union curves relative to the congruence (2.4) on quadrics form an hexagonal three - web.*

Application II. 1.2. Pseudo - Spherical surfaces. Let a pseudo - spherical surface of curvature $-\frac{1}{a^2}$ be defined in terms of isothermal conjugate parameters. Then the fundamental quantities can be chosen as

$$g_{11} = a^2 \cos^2\omega \quad , \quad g_{12} = 0 \quad , \quad g_{22} = a^2 \sin^2\omega$$

and

$$d_{11} = - d_{22} = - a \sin \omega \cdot \cos \omega \quad , \quad d_{12} = 0$$

where $\omega = \omega(u^1, u^2)$ is a function which must satisfy the equation

$$\text{(2.13)} \qquad \frac{\partial^2 \omega}{\partial u^1 \, \partial u^1} - \frac{\partial^2 \omega}{\partial u^2 \, \partial u^2} = \sin \omega \cdot \cos \omega \qquad [^{10}, 286] .$$

For the surface S, since $g_{12} = d_{12} = 0$ and $K = \text{const.}$, according to theorem **II. 3.**, the necessary and sufficient condition for any three families of union curves relative to the congruence (2.4) to form an hexagonal three - web on S is

$$(2.14) \qquad \left(\log \frac{g_{11}\sqrt{g_{11}}}{d_{11}\sqrt{g_{22}}}\right)_{,\,12} = \{\log | - a \cdot \operatorname{cotg}^2 \omega \, | \}_{,\,12} = 0$$

and hence

$$(\log \operatorname{tg} \omega)_{,\,12} = 0 \quad \text{or} \quad \operatorname{tg} 2\omega = 2 \, \frac{\omega_{,\,1} \cdot \omega_{,\,2}}{\omega_{,\,12}} \, .$$

On the other hand, using equation (2.13) and making the necessary calculation, equation (2.14) takes the form

$$(2.15) \qquad (\omega_{,\,11} - \omega_{,\,22})^2 \, (\omega^2_{,\,12} + 4\omega^2_{,\,1} \cdot \omega^2_{,\,2}) - \omega^2_{,\,1} \cdot \omega^2_{,\,2} = 0 \, .$$

Thus the following theorem is obtained :

Theorem II. 6. *The necessary and sufficient condition for any three families of union curves relative to the congruence* (2.4) *to form an hexagonal three - web on a pseudo - spherical surface is that* ω *be a function of* u^1, u^2 *satisfying the differential equation*

$$(\omega_{,\,11} - \omega_{,\,22})^2 \, (\omega^2_{,\,12} + 4\omega^2_{,\,1} \cdot \omega^2_{,\,2}) - \omega^2_{,\,1} \cdot \omega^2_{,\,2} = 0 \, .$$

Application II. 1.3. Surfaces of revolution. From the parametric equations of the surfaces of revolution

$$x^1 = A(u^1) \cos u^2 \quad , \quad x^2 = A(u^1) \sin u^2 \quad , \quad x^3 = B(u^1)$$

we find that

$$g_{11} = [A'(u^1)]^2 + [B'(u^1)]^2 \quad , \quad g_{12} = 0 \quad , \quad g_{22} = [A(u^1)]^2$$

and

$$d_{11} = \frac{A'(u^1) \cdot B'(u^1) - A''(u^1) \cdot B'(u^1)}{\sqrt{A'^2 + B'^2}} \; , \; d_{12} = 0 \; , \; d_{22} = \frac{A(u^1) \cdot B'(u^1)}{\sqrt{A'^2 + B'^2}} \, .$$

Since

$$g_{11} = g_{11}(u^1) \quad , \quad g_{22} = g_{22}(u^1) \quad , \quad d_{11} = d_{11}(u^1) \quad , \quad d_{22} = d_{22}(u^1)$$

$$g_{12} = d_{12} = 0$$

these surfaces satisfy all the conditions of theorem **II. 4**. Thus the following theorem is obtained :

Theorem II. 7. *Any three families of union curves relative to the congruence* (2.4) *on the non-developable surfaces of revolution form an hexagonal three-web.*

Since $g_{12} = d_{12} = 0$ for these surfaces, the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma} (\alpha, \beta, \gamma = 1,2)$ take the form :

$$\Gamma^1_{11} = \frac{A' \cdot A'' + B' \cdot B''}{A'^2 + B'^2} \quad , \quad \Gamma^2_{12} = \frac{A'}{A} \quad , \quad \Gamma^1_{22} = \frac{A \cdot A'}{A'^2 + B'^2}$$

$$\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^2_{22} = 0$$

and therefore, the congruence (2.4), which is formed by the intersections of the osculating planes of the two lines of curvature cutting each other at that particular point of the surface is

$$z^i = x^i + t \left\{ - \frac{A'}{B' \sqrt{A'^2 + B'^2}} \, x^i{}_{,1} + X^i \right\}$$

and the differential equation of the union curves relative to this congruence is

$$\frac{d^2 u^2}{du^1 \, du^1} + \left(2 \frac{A'}{A} - \frac{B''}{B'} \right) \frac{du^2}{du^1} = 0 \quad , \quad (B' \neq 0).$$

By the first integration of this differential equation we obtain

$$u^2 = c_1 \int \frac{B'(u^1)}{[A(u^1)]^2} \, du^1 \quad , \quad (c_1 = \text{const.}).$$

Application II. 1.4. Isothermic surfaces which are applicable to surfaces of revolution. It is well known that, when the linear element of a surface reducible to the form

$$ds^2 = \nu(u^\alpha) \; [(du^1)^2 + (du^2)^2]$$

where $\nu = \nu(u^\alpha)$, $(\alpha = 1,2)$, the surface is applicable to a surface of revolution. At the same time, if the lines of curvature of this surface are parametric and the surface is isothermic (this kind of surfaces truly exist, for example the limit surfaces of a group of applicable surfaces are isothermic [8, 389]) so we find that

$$g_{11} = g_{22} = \nu(u^\alpha) \quad , \quad g_{12} = d_{12} = 0 .$$

Thus, these surfaces satisfy all the conditions of theorem II. 4. and the following theorem is obtained :

Theorem II. 8. *Any three families of union curves relative to the congruence* (2.4) *on a non developable isothermic surface which is applicable to a surface of revolution, form an hexagonal three - web.*

If for these surfaces we have $g_{11} = g_{22} = \nu(u^\alpha)$, then the Christoffel Symbols take the form

$$\begin{aligned} &\Gamma^\beta_{\alpha\alpha} = \Gamma^\alpha_{\alpha\beta} = \Gamma^\beta_{\beta\beta} = 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad (\alpha, \beta = 1,2 \ ; \ \alpha \neq \beta) \\ &\Gamma^\alpha_{\alpha\alpha} = \Gamma^\beta_{\alpha\beta} = - \Gamma^\alpha_{\beta\beta} = (\log \sqrt{g_{\alpha\alpha}})_{,\alpha} \end{aligned}$$

From the Codazzi and Gauss equations, we find that

$$d_{\alpha\alpha} = d_{\alpha\alpha}(u^\alpha) \quad \text{and} \quad d_{\beta\beta} = d_{\beta\beta}(u^\alpha) .$$

Thus, the congruence (2.4) takes the form

$$z^i = x^i + t \left\{ - \frac{(\log \sqrt{g_{\alpha\alpha}})_{,\alpha}}{d_{\beta\beta}} x^i_{,\alpha} + X^i \right\} \quad , \quad (\alpha \neq \beta)$$

and the differential equation (2.11) of the union curves relative to this congruence is

$$du^\alpha - c \frac{g_{\beta\beta}(u^\alpha)}{d_{\beta\beta}(u^\alpha)} du^\beta = 0 \quad , \quad (\alpha \neq \beta) . \tag{2.16}$$

By using the normal curvatures $r^\beta = \dfrac{d_{\beta\beta}}{g_{\beta\beta}}$, $(\beta = 1,2)$ of the parametric lines $du^\alpha = 0$, the equation (2.16) takes the form

$$du^\beta - c_1 \, r^\beta (u^\alpha) \, du^\alpha = 0 .$$

Thus, the equation of union curves relative to the congruence (2.4) on isothermic surfaces which are applicable to surfaces of revolution, is

$$u^{\beta} = c_1 \int r^{\beta}(u^{\alpha})\, du^{\alpha} \quad , \quad (\alpha, \beta = 1,2 \; ; \; \alpha \neq \beta)$$

where $g_{11} = g_{22} = \nu(u^{\alpha})$.

II. 2. Union curves on non - developable isothermic surfaces. Let S be an isothermic surface and let it be referred to its lines of curvature. Since the lines of curvature form an isothermal system on an isothermic surface, the coefficients of the first and second fundamental forms are of the form

$$g_{11}(u^1, u^2) = g_{22}(u^1, u^2) \quad , \quad g_{12} = d_{12} = 0 \, .$$

According to the theorem II. 3., since the Gaussian curvature is of the form $K = U_1(u^1) \cdot U_2(u^2) \neq 0$, the necessary and sufficient condition for any three families of union curves relative to the congruence (2.4) to form an hexagonal three - web on S is

$$\frac{g_{\alpha\alpha}}{d_{\alpha\alpha}} = f_1(u^1) \cdot f_2(u^2) \quad , \quad (\text{for a certain } \alpha) \, .$$

By using the normal curvatures r^{α} of the parametric lines $du^{\beta} = 0$, we obtain

$$r^{\alpha} = f_1^*(u^1) \cdot f_2^*(u^2)$$

and so the following theorem is obtained :

Theorem II. 9. *A sufficient condition for any three families of union curves relative to the congruence* (2.4) *to form an hexagonal three - web on the isothermic surfaces is*

$$r^{\alpha} = f_1^{\alpha}(u^1) \cdot f_2^{\alpha}(u^2) \neq 0 \quad , \quad (\alpha = 1,2)$$

where r^{α} are the normal curvatures of the parametric lines $du^{\beta} = 0$.

Application II. 2.1. Dupin's Cyclides. Since the lines of curvature on a Dupin Cyclide form an isothermally orthogonal net, by referring it to its lines of curvature, we obtain

$$g_{11} = g_{22} \quad , \quad g_{12} = d_{12} = 0 \, .$$

On the other hand, since Dupin's Cyclides are characterized by the conditions [11, 141] .

$$r^{\alpha}_{\alpha} = 0 \quad , \quad (\alpha = 1{,}2)$$

where r^{α}_{1} and r^{α}_{2} are the invariant derivatives of r^{α}, calculated along the $du^2 = 0$ and $du^1 = 0$ respectively, [1)]

we obtain

$$r^1 = r^1(u^2) \quad \text{and} \quad r^2 = r^2(u^1)\,.$$

So, these surfaces satisfy all the conditions of the theorem **II. 1.** and the following theorem is obtained :

Theorem II. 10. *Any three families of union curves relative to the congruence* (2.4) *on Dupin's Cyclides, form an hexagonal three - web.*

According to (2.11), the differential equation of the union curves relative to the congruence (2.4) on Dupin's Cyclides is

$$r^1(u^2)\, du^2 - c\, r^2(u^1)\, du^1 = 0 \quad , \quad (c = \text{const.})\,.$$

II. 3. Union curves on parallel surfaces. Let

$$S: \quad x^i = x^i(u^1,\, u^2)$$

be a surface and let

$$S^* : \quad x^{*i}(u^1,\, u^2) = x^i(u^1,\, u^2) + a \cdot X^i(u^1,\, u^2)$$

where X^i denote the direction cosines of the normal to S and a is a constant, be a surface parallel to S. The magnitudes of the first and second order for the parallel surface are (relative to those of S)

$$g^*_{\alpha\alpha} = (ar^{\alpha} - 1)^2\, g_{\alpha\alpha} \quad , \quad g^*_{\alpha\beta} = 0$$

and $\qquad\qquad\qquad\qquad\qquad (\alpha,\ \beta = 1{,}2\ ;\ \alpha \neq \beta)$

$$d^*_{\alpha\alpha} = -\,(ar^{\alpha} - 1)\, d_{\alpha\alpha} \quad , \quad d^*_{\alpha\beta} = 0\,.$$

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1)
$$r^{\alpha}_{\beta} = \frac{\dfrac{\partial r^{\alpha}}{\partial u^{\beta}}}{\sqrt{g_{\beta\beta}}} = \frac{r^{\alpha}{}_{,\beta}}{\sqrt{g_{\beta\beta}}} \quad , \quad (\alpha,\ \beta = 1{,}2)$$

Thus

$$r^{\alpha *} = -\frac{r^{\alpha}}{(ar^{\alpha} - 1)} \quad , \quad (\alpha = 1,2)$$

and

$$K = \frac{r^1 \cdot r^2}{(ar^1 - 1)\,(ar^2 - 1)} \,.$$

Application II. 3.1. Parallel surfaces to Dupin's Cyclides. For the parallel surfaces to Dupin's Cyclides, we have

$$r^{1*} = -\frac{r^1(u^2)}{(ar^1 - 1)} = r^{1*}(u^2)$$

$$r^{2*} = -\frac{r^2(u^1)}{(ar^2 - 1)} = r^{2*}(\mathrm{u}^1)$$

and

$$\frac{\overset{*}{g}_{\alpha\alpha} \sqrt{\overset{*}{g}_{\alpha\alpha}}}{\overset{*}{d}_{\alpha\alpha} \sqrt{\overset{*}{g}_{\beta\beta}}} = -\frac{(ar^{\alpha} - 1)^2}{(ar^{\beta} - 1)\, r^{\alpha}} = -\frac{[ar^{\alpha}(u^{\beta}) - 1]^2}{r^{\alpha}(u^{\beta})} \cdot \frac{1}{[ar^{\beta}(u^{\alpha}) - 1]}$$

therefore these surfaces satisfy ail the conditions of the theorem II. 3.

Application II. 3.2. Parallel surfaces to surfaces of revolution and to isothermic surfaces which are applicable to surfaces of revolution. Both surfaces, the surfaces of revolution and isothermic surfaces which are applicable to the surfaces of revolution, have the following fundamental magnitudes of the first and second order

$$g_{11} = g_{11}(u^1) \quad , \quad g_{22} = g_{22}(u^1) \quad , \quad g_{12} = 0$$

and

$$d_{11} = d_{11}(u^1) \quad , \quad d_{22} = d_{22}(u^1) \quad , \quad d_{12} = 0 \,.$$

Thus, for the parallel surfaces to these surfaces, g_{11} and g_{22} are the functions of u^1 alone, so we see that these parallel surfaces satisfy all the conditions of the theorem II. 4. and finally the following theorem is obtained :

Theorem II. 11. *Any three families of union curves, relative to the congruence* (2.4), *on surfaces parallel to Dupin's Cyclides, to surfaces of revolution and to isothermic surfaces which are applicable to surfaces of revolution form an hexagonal three - web.*

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ÖZET

Bu çalışmada, bir S yüzeyinin her noktasından geçen iki eğrilik çizgisinin oskülatör düzlemlerinin ara kesit doğrularının oluşturduğu kongrüansa bağlı birleşim eğrilerinden herhangi üç ailenin altıgen doku teşkil etme şartları araştırılmakta ve özellikle, eğrilik çizgilerine nisbet edildiğinde Gauss Eğriliği $K = U_1(u^1) \,.\, U_2(u^2) \neq 0$ şeklinde çarpanlarına ayrılabilen yüzeyler için kesin sonuçlar verilmektedir.

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