ON THE SUBCLASSES U_m IN MAHLER'S CLASSIFICATION OF THE TRANSCENDENTAL NUMBERS *)

KÂMİL ALNIAÇIK

In this paper integral and rational combinations with algebraic coefficients of a strong Liouville number are studied and shown that they belong to the Mahler subclass U_m, where m is the degree of the algebraic number field determined by these coefficients. Thus a new proof is obtained for the fact which was first proved by LEVEQUE in 1953, that no Mahler subclass $U_m (m = 1, 2, \ldots)$ is empty. In the case of integral combinations an analogous result for Hensel's field of p - adic numbers is given.

CHAPTER I

Mahler's classification. We shall be concerned with polynomials $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0, \; a_0 \neq 0$, with rational integer coefficients. The height $H(P)$ of P is defined by $H(P) = \max (|a_n|, |a_{n-1}|, \ldots, |a_0|)$.

Given an arbitrary complex number ξ, for any real number $H \geqslant 1$ and a natural number n Mahler puts

$$w_n(H, \xi) = \min_{\substack{\deg P \leqslant n \\ H(P) \leqslant H \\ P(\xi) \neq 0}} |P(\xi)| .$$

As $H \geqslant 1$, one may take $P(x) = 1$, and hence we have $0 < w_n(H, \xi) \leqslant 1$. If either n or H increases, $w_n(H, \xi)$ will not increase. Next, MAHLER puts

*) This paper is an English translation of the substance of a doctoral dissertation accepted by the Faculty of Science of the University of Istanbul in September 1978. I am grateful to Prof. Dr. Orhan Ş. İÇEN for his valuable help and encouragement in all stages of this work.

$$w_n(\xi) = \limsup_{H \to \infty} (-\log w_n(H, \xi)/\log H)$$

and

$$w(\xi) = \limsup_{n \to \infty} \frac{w_n(\xi)}{n}.$$

By what we have said above, $w_n(\xi)$ as a function of n is nondecreasing. One has always $0 \leqslant w_n(\xi) \leqslant +\infty$ and $0 \leqslant w(\xi) \leqslant +\infty$.

If $w_n(\xi) = +\infty$ for some integer n, let $\mu(\xi)$ be the smallest such integer; if $w_n(\xi) < +\infty$ for every n, put $\mu(\xi) = \infty$.

MAHLER calls the number ξ an

A - number if $w(\xi) = 0$, $\mu(\xi) = \infty$,

S - number if $0 < w(\xi) < \infty$, $\mu(\xi) = \infty$,

T - number if $w(\xi) = \infty$, $\mu(\xi) = \infty$,

U - number if $w(\xi) = \infty$, $\mu(\xi) < \infty$

(See MAHLER [8]). A - numbers are identical with algebraic numbers, whereas the transcendental numbers are distributed into the three classes S, T, U. Let ξ be a U - number such that $\mu(\xi) = m$ and let U_m denote the set of all such numbers. It is obvious that for every natural m, the class U_m is a subclass of U and $U = \bigcup_{m=1}^{\infty} U_m$. Moreover we have $U_m \cap U_n = \phi$ if $m \neq n$. (For the subclasses U_m see LEVEQUE [6]).

We shall now collect some lemmas which will be used in chapters I and II. Those which are taken from elsewhere will be given without proof, but with reference to their sources.

Lemma 1. *Let α be an algebraic number of degree s and let $P(x)$ be an arbitrary polynomial of degree n with integral coefficients. If $P(\alpha) \neq 0$, then the relation*

$$|P(\alpha)| \geqslant \frac{1}{[(n+1)H]^{s-1}[(s+1)h]^n}$$

holds, where H *is the height of* $P(x)$ *and* h *is the height of the minimal polynomial of the algebraic number* α*, respectively.* (R. GÜTING [3], Th. 5).

Lemma 2. *Let* z_1, z_2 *be two complex numbers and* $P(x)$ *be a polynomial with arbitrary complex coefficients. Then there is a complex number* η *with* $0 \leqslant |\eta| \leqslant 1$ *and a complex number* σ *on the segment* $\overline{z_1 z_2}$ *such that* $P(z_1) - P(z_2) = \eta(z_1 - z_2) P'(\sigma)$*, where* $P'(x)$ *denotes the derivative of* $P(x)$. (See BIEBERBACH [1], p. 116).

Lemma 3. *Let* $\alpha_1, \ldots, \alpha_k (k \geqslant 1)$ *be algebraic numbers which belong to an algebraic number field* K *of degree* g*, and let* $F(y, x_1, \ldots, x_k)$ *be a polynomial with rational integral coefficients and with degree at least one in* y*. If* η *is an algebraic number such that* $F(\eta, \alpha_1, \ldots, \alpha_k) = 0$*, then the degree of* $\eta \leqslant dg$*, and*

$$h_\eta \leqslant 3^{2dg + (l_1 + \ldots + l_k) g} \cdot H^g \cdot h_{\alpha_1}^{l_1 g} \ldots h_{\alpha_k}^{l_k g},$$

where h_η *is the height of* η*,* H *is the maximum of the absolute values of the coefficients of* F*,* $l_i (i = 1, \ldots, k)$ *is the degree of* F *in* $x_i (i = 1, \ldots, k)$*,* d *is the degree of* F *in* y*, and* h_{α_i} *is the height of* $\alpha_i (i = 1, \ldots, k)$. (See O. Ş. İÇEN [4]).

Lemma 4. *Let* α_1, α_2 *be two algebraic numbers with different minimal polynomials. Then we have*

$$|\alpha_1 - \alpha_2| \geqslant \frac{1}{2^{\max(n_1, n_2) - 1} [(n_1 + 1) h_1]^{n_2} [(n_2 + 1) h_2]^{n_1}},$$

where n_1, n_2 *are the degrees and* h_1, h_2 *the heights of* α_1, α_2 *respectively.* (See GÜTING [1], Th. 7).

Lemma 5. *Let* $\alpha_0, \ldots, \alpha_k$; $\beta_0, \ldots, \beta_l (k \geqslant 0,\ l \geqslant 0,\ \max\ (k, l) > 0,$ $\alpha_k \neq 0,\ \beta_l = 1)$ *be algebraic numbers with* $[Q(\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_l) : Q] = m.$[1]

[1] Here Q denotes as usual the field of rational numbers.

If the polynomials $C(x) = \alpha_0 + \dots + \alpha_k x^k$, $D(x) = \beta_0 + \dots + \beta_l x^l$ *are relatively prime, then for* $x \in Q$ *the algebraic number* $\theta_x = \dfrac{C(x)}{D(x)}$ *is a primitive element of the field* $Q(\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l)$ *except for only finitely many values of* x.

Proof. Let $\alpha_i^{(\nu)}$, $\beta_j^{(\nu)}$ $(\nu = 1, \dots, m)$ be the field conjugates of α_i, β_j respectively. Take as usual $\alpha_i^{(1)} = \alpha_i$, $\beta_j^{(1)} = \beta_j$ $(i = 0, \dots, k\ ;\ j = 0, \dots, l)$ and put $K = Q(\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l)$. From the outset we exclude the values of x which satisfy $C(x) = 0$ or $D(x) = 0$, if any, which constitute a finite set.

Now we have two cases according as $m = 1$ or $m > 1$:

a) Let $m = 1$. Then the algebraic numbers $\alpha_i (i = 0, \dots, k)$, $\beta_j (j = 0, \dots, l)$ are rational numbers and the lemma is obvious.

b) Let $m > 1$. If θ_{x_0} is not a primitive element of the field K, then there is a field conjugate $\theta_{x_0}^{(\nu)}$ with $\nu_0 \neq 1$, for which the relation

$$\theta_{x_0} = \theta_{x_0}^{(\nu_0)} \tag{1}$$

holds.

From (1) we obtain

$$C(x_0)\, D^{(\nu_0)}(x_0) = C^{(\nu_0)}(x_0)\, D(x_0), \text{ where we have put} \tag{2}$$

$$C^{(\nu)}(x) = \alpha_0^{(\nu)} + \dots + \alpha_k^{(\nu)} x^k, \quad D^{(\nu)}(x) = \beta_0^{(\nu)} + \dots + \beta_l^{(\nu)} x^l.$$

If (1) and consequently (2) were true for infinitely many values x_0 of x, we would have identically

$$C(x)\, D^{(\nu_0)}(x) = C^{(\nu_0)}(x)\, D(x). \tag{3}$$

As $C(x)$ is relatively prime to $D(x)$, it must divide $C^{(\nu_0)}(x)$. But as $C^{(\nu_0)}(x)$ is of the same degree as $C(x)$, there must exist a complex constant $\lambda \neq 0$ such that $C^{(\nu_0)}(x) = \lambda\, C(x)$. This with (3) would give $D^{(\nu_0)}(x) = \lambda\, D(x)$.

But we have for the leading coefficients of $D(x)$ and $D^{(\nu_0)}(x)$, $\beta_l = 1$ and $\beta_l^{(\nu_0)} = 1$ respectively, so the comparison of the leading coefficients on both sides of $D^{(\nu_0)}(x) = \lambda\, D(x)$ would give $\lambda = 1$, and consequently $C(x) = C^{(\nu_0)}(x)$, $D(x) = D^{(\nu_0)}(x)$, whence we would obtain

$$(4)\qquad \begin{cases} \alpha_i = \alpha_i^{(\nu_0)} & (i = 0,\dots k) \\ \beta_j = \beta_j^{(\nu_0)} & (j = 0,\dots l). \end{cases}$$

But this would lead us to a contradiction as follows :

As $m > 1$, $Q(\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l)$ is a proper extension of Q, so there exists a primitive element ζ of this extension of degree $m > 1$ over Q. We have

$$(5)\qquad \zeta = R(\alpha_0, \dots \alpha_k, \beta_0, \dots \beta_l)$$

and

$$(6)\qquad \begin{cases} \alpha_i = S_i(\zeta) & (i = 0,\dots, k), \\ \beta_j = T_j(\zeta) & (j = 0,\dots, l), \end{cases}$$

where R, S, T denote rational functions of their arguments with coefficients from Q. If the conjugates of ζ are denoted by $\zeta^{(\nu)}(\nu = 1,\dots, m)$, with $\zeta^{(1)} = \zeta$, which are all different, then the field conjugates of α_i, β_l are

$$(7)\qquad \begin{cases} \alpha_i^{(\nu)} = S_i(\zeta^{(\nu)}) & (i = 0,\dots k), \\ \beta_j^{(\nu)} = T_j(\zeta^{(\nu)}) & (j = 0,\dots l) \end{cases}$$

respectively, which satisfy

$$(8)\qquad \zeta^{(\nu)} = R(\alpha_0^{(\nu)}, \dots, \alpha_k^{(\nu)}, \beta_0^{(\nu)}, \dots, \beta_l^{(\nu)}) \qquad (\nu = 1,2,\dots m).$$

Now (4), (5), (8) would give us

(9) $$\zeta = \zeta^{(\nu_0)},$$

which would contradict that ζ is a primitive element.

Definition. Let ξ be a Liouville number with convergents $\frac{h_n}{k_n}$ $(n=0,1,\ldots)$ in its regular continued fraction expansion and let $|k_n\,\xi - h_n| := k_n^{-s_n}$. We shall say that ξ is strong or weak according as $\liminf_{n\to\infty} s_n$ is infinite or finite. (LE VEQUE [1]).

(For any Liouville number we have of course $\limsup_{n\to\infty} s_n = +\infty$).

Theorem I. *Let $\alpha_0, \ldots, \alpha_k$ $(k \geqslant 1,\ \alpha_k \neq 0)$ be algebraic numbers such that $[Q(\alpha_0, \ldots, \alpha_k) : Q] = m$, and let $C(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_k x^k$. If ξ is a strong Liouville number, then the number $C(\xi) = \gamma$ belongs to U_m.*

Proof. Let the convergents of the regular continued fraction expansion of the Liouville number ξ be $\frac{p_n}{q_n}$ $(n = 0,1,\ldots)$. Since the Liouville number ξ is strong, for the sequence $\omega(n) = \omega_n$ defined by $\left|\xi - \frac{p_n}{q_n}\right| = q_n^{-\omega(n)} (n=0,1,\ldots)$ we have $\liminf_{n\to\infty} \omega_n = +\infty$. Then we have

(10) $$\xi = \frac{p_n}{q_n} + \varepsilon_n\, q_n^{-\omega(n)} \quad (\varepsilon_n = \pm 1\,,\ \ n = 0,1,\ldots).$$

Now we apply Lemma 2 with $P(z) = C(z)$, $z_1 = \xi$, $z_2 = \frac{p_n}{q_n}$ $(n = 0,1,\ldots)$, and we get

(11) $$C(\xi) - C\left(\frac{p_n}{q_n}\right) = \eta_1 \left(\xi - \frac{p_n}{q_n}\right) C'(\theta_n) \qquad (n = 0,1,\ldots),$$

where η_1 is a complex number with $0 \leqslant |\eta_1| \leqslant 1$ and θ_n is a real number in the interval $\xi \cdots \frac{p_n}{q_n}$. Since $\lim_{n\to\infty} \frac{p_n}{q_n} = \xi$, there is a natural number N_0 such that

$$(12)\qquad \left|\frac{p_n}{q_n}\right| < 2\,|\xi|\,, \quad 0 \leqslant |\theta_n| < 2\,|\xi| \ \text{ for every } n > N_0\,.$$

Using this, we obtain

$$(13)\qquad |C'(\theta_n)| < k^2 \cdot \max_{i=0}^{k} (|\alpha_i|) \cdot \max\,[1, (2\,|\xi|)^k] = c_1 \qquad (n > N_0),$$

where $c_1 > 0$ is independent of n. [1)]

For $n > N_0$ let $P_n(x)$ denote the minimal polynomial of the algebraic number $C\left(\frac{p_n}{q_n}\right)$, and let $H(P_n)$ be the height of $P_n(x)$.

Applying Lemma 2 with $P(z) = P_n(z)$, $z_1 = C(\xi)$, $z_2 = C\left(\frac{p_n}{q_n}\right)$ $(n > N_0)$ we have

$$(14)\qquad P_n(\gamma) - P_n\left(C\left(\frac{p_n}{q_n}\right)\right) = \eta_2\left(\gamma - C\left(\frac{p_n}{q_n}\right)\right) P_n'(\widetilde{\theta}_n) \qquad (n > N_0),$$

where η_2 is a complex number with $0 \leqslant |\eta_2| \leqslant 1$ and $\widetilde{\theta}_n$ is a point on the segment $\overline{\gamma \cdot C\left(\frac{p_n}{q_n}\right)}$. Hence there exists a real number t with $0 \leqslant t \leqslant 1$, such that

$$(15)\qquad \widetilde{\theta}_n = (1 - t)\,\gamma + t\,C\left(\frac{p_n}{q_n}\right) \qquad (n > N_0).$$

On the other hand we have by (12)

$$\left|C\left(\frac{p_n}{q_n}\right)\right| \leqslant (k+1) \max_{i=0}^{k} (|\alpha_i|) \cdot \max\,[1, (2\,|\xi|)^k] = c_2 \qquad (c_2 > 0),$$

and using this in the relation (15) we obtain

$$(16)\qquad |\widetilde{\theta}_n| \leqslant |\gamma| + c_2 = c_3 \qquad (n > N_0), \quad (c_3 > 0).$$

[1)] Here and in the sequel $c_1, c_2, \ldots$ will denote positive real numbers independent of n.

Now we know that $[K:Q]=m$, hence analogously to (13) we see that

$$(17) \qquad |P_n'(\widetilde{\theta_n})| \leqslant m^2 \cdot \max(1, c_3^m) \cdot H(P_n) \qquad (n > N_0).$$

It follows from the definition of the polynomial $P_n(x)$ that $P_n\left(C\left(\frac{p_n}{q_n}\right)\right)=0$. Hence using this in (14) and combining the relations (10), (11), (13) and (17), we obtain

$$|P_n(\gamma)| \leqslant c_1 \cdot m^2 \cdot \max(1, c_3^m) \cdot H(P_t) \cdot q_n^{-\omega(n)} \qquad (n > N_0),$$

and so putting $c_4 = c_1 \cdot m^2 \cdot \max(1, c_3^m)$:

$$(18) \qquad 0 < |P_n(\gamma)| \leqslant c_4 \cdot q_n^{-\omega(n)} \cdot H(P_n) \qquad (n > N_0).$$

($P_n(\gamma) = P_n(C(\xi))$ is not zero, since ξ is a transcendental number.)

Now, we shall give an upper bound for the height of $P_n(x)$. Put

$$(19) \qquad \gamma_n = C\left(\frac{p_n}{q_n}\right) \qquad (n > N_0),$$

or what is the same thing

$$\gamma_n q_n^k - \alpha_0 q_n^k - \alpha_1 p_n q_n^{k-1} - \dots - \alpha_k p_n^k = 0.$$

We see from (15) that, the value of the polynomial

$$F(y, x_0, x_1, \dots x_k) = q_n^k y - q_n^k x_0 - p_n q_n^{k-1} x_1 - \dots - p_n^k x_k$$

is zero for $y = \gamma_n$, $x_i = \alpha_i (i = 0, \dots, k)$.

Therefore we can use Lemma 3 with $d = 1$, $l_i = 1$ $(i = 0, \dots, k)$, $g = m$,

$$H \leqslant \max[1, (2|\xi|)^k]\, q_n^k,$$

and we obtain

$$H(P_n) \leqslant 3^{(k+3)m} \cdot \{\max[1, (2|\xi|)^k]\}^m \cdot q_n^{k \cdot m} \cdot h_{a_0}^m \dots h_{a_k}^m,$$

or putting $c_5 = 3^{(k+3)m} \cdot \{\max[1, (2|\xi|)^k]\}^m \cdot h_{a_0}^m \dots h_{a_k}^m$,

$$(20) \qquad H(P_n) \leqslant c_5 \cdot q_n^{km} \qquad (n > N_0).$$

Since c_5 is independent of n, there is a natural number N_1 for which the relation

$$(21) \qquad H(P_n) < q_n^{km+1}$$

holds for $n > \max(N_0, N_1)$.

Finally, combining the relations (18) and (21) we have

$$(22) \qquad |P_n(\gamma)| \leqslant \frac{c_4 \, H(P_n)}{q_n^{\omega(n)}} \leqslant \frac{c_4}{\left(H(P_n)\right)^{\frac{\omega(n)}{km+1}-1}} \qquad (n > \max(N_0, N_1)).$$

As γ was taken as a Liouville number we have $\limsup_{n\to\infty} \omega(n) = +\infty$, so that we can choose a subsequence $\omega(n_j)$ with $\lim_{n_j \to \infty} \omega(n_j) = +\infty$. (22) will give for this subsequence

$$(23) \qquad 0 < |P_{n_j}(\gamma)| \leqslant \frac{c_4}{H(P_{n_j})^{\frac{\omega(n_j)}{km+1}-1}} \qquad (n_j > \max(N_0, N_1)).$$

Now the sequence of heights $\{H(P_{n_j})\}$ must contain a subsequence $\{H(P_{n_{j_k}})\}$ tending to $+\infty$. For otherwise $\{H(P_{n_j})\}$ would be bounded from above and as the degrees of the polynomials $P_{n_j}(x)$ are also bounded ($\leqslant m$), the sequence of polynomials $\{P_{n_{j_l}}(x)\}$ would contain only a finite number of different polynomials, therefore it would have at least one identical subsequence. Let this be denoted with $\{P_{n_{j_l}}(x)\}$, where $P_{n_{j_l}}(x) = \widetilde{P}(x)$ say, for all l.

But we had $P_{n_{j_l}}\left(C\left(\frac{p_{n_{j_l}}}{q_{n_{j_l}}}\right)\right) = 0$ for ail l, which would give us

$$\widetilde{P}\left(C\left(\frac{p_{n_{j_l}}}{q_{n_{j_l}}}\right)\right) = 0 \qquad (l = 1,2,\ldots).$$

By letting $l \to \infty$ we obtain $\widetilde{P}(C(\xi)) = 0$, which would mean that ξ is algebraic, in contradiction to its being a Liouville number. Thus we obtain

$$(24) \qquad 0 < |P_{n_{j_k}}(\gamma)| \leqslant \frac{c_4}{\left(H(P_{n_{j_k}})\right)^{\frac{\omega(n_{j_k})}{km+1}-1}} \qquad (n_{n_{j_k}} > \max(N_0, N_1)),$$

with $\lim\limits_{k\to\infty} H(P_{n_{j_k}}) = +\infty$ and $\lim\limits_{k\to\infty} \omega(n_{j_k}) = +\infty$. Since the degree of $P_{n_{j_k}}(x) \leqslant m$, the relation (24) shows that

$$(\ast) \qquad \mu(\gamma) \leqslant m\,.$$

We shall complete the proof by showing the opposite inequality $\mu(\gamma) \geqslant m$, and for this we shall distinguish two cases according as $m = 1$ or $m > 1$.

I — *In the case* $m = 1$, by definition of $\mu(\gamma)$ we have $\mu(\gamma) \geqslant 1$, so together with $(\ast)$ for $m = 1$, we obtain $\mu(\gamma) = 1$.

II — *Suppose that* $m > 1$. Let $P(x)$ be a polynomial of degree l $(0 < l \leqslant m - 1)$ with integral coefficients, and let $H(P)$ denote the height of $P(x)$. Analogously to (14), by Lemma 2 we have

$$(25) \qquad P(\gamma) - P(\gamma_n) = \eta_3(\gamma - \gamma_n)\, P'(\approx\!\theta_n), \qquad (n > \max(N_0, N_1)),$$

where η_3 and $\approx\!\theta_n$ $(n > \max(N_0, N_1))$ are complex numbers such that

$$0 \leqslant |\eta_3| \leqslant 1, \quad |\approx\!\theta_n| \leqslant c_3 \qquad (n > \max(N_0, N_1)).$$

Hence we can write

$$|P'(\widetilde{\widetilde{\theta}}_n)| \leqslant m^2 \cdot \max(1, c_3^m) \cdot H(P), \qquad n > \max(N_0, N_1), \tag{26}$$

and using this and (11) in (25), we obtain

$$|P(\gamma)| \geqslant |P(\gamma_n)| - c_4\, q_n^{-\omega(n)} \cdot H(P), \qquad (n > \max(N_0, N_1)). \tag{27}$$

On the other hand, by Lemma 5, there is an integer N_2 such that, if $n > N_2$ then the degree of the algebraic number γ_n is equal to m. Since $l < m$, we can use Lemma 1 with

$$\alpha = \gamma_n\,(n > \max(N_0, N_1, N_2)),\ s = m,\ \mathrm{n} = l,\ h = H(P_n)$$

and we get

$$|P(\gamma_n)| \geqslant \frac{1}{(l+1)^{m-1}\,(m+1)^l\, H(P)^{m-1}\, H(P_n)^l}\,. \tag{28}$$

Using (20) in (28) and putting $(m+1)^{1-m}\, m^{1-m}\, c_5^{1-m} = c_6$ we have

$$|P(\gamma_n)| \geqslant \frac{c_6}{H(P)^{m-1}\, q_n^{km(m-1)}} \qquad (n > \max(N_0, N_1, N_2)), \tag{29}$$

and combining the relations (27) and (29)

$$|P(\gamma)| \geqslant \frac{c_6}{H(P)^{m-1}\, q_n^{km(m-1)}} - \frac{c_4\, H(P)}{q_n^{\omega(n)}}\,. \tag{30}$$

It follows from well known properties of continued fractions that if $\left|\xi - \frac{p_n}{q_n}\right| = q_n^{-\omega(n)}$, then

$$q_n^{\omega(n)} \geqslant q_{n+1} > q_n^{\omega(n)-2} \qquad (n > N_3), \tag{31}$$

where N_3 is a suitable natural number.

On the other hand, by assuming that ξ is a strong Liouville number, there is a natural number N_4 such that

$$\omega(n) > km(m-1)\,[(km+1)(m-1)+2] + m + 1 \qquad (n > N_4). \tag{32}$$

Now suppose that the polynomial $P(x)$ satisfies the condition

$$H(P) > \max\left(q_{\nu_0}, \frac{2c_4}{c_6}\right), \tag{33}$$

where ν_0 is a fixed index satisfying $\nu_0 > \max(N_0, N_1, N_2, N_3, N_4)$. It is clear that, there exists a natural number $\nu \geqslant \nu_0$ for every polynomial $P(x)$ which satisfies (33), such that

$$q_\nu \leqslant H(P) < q_{\nu+1}. \tag{34}$$

From (31) and (32) we see that the inequality

$$q_\nu < q_{\nu+1}^{\frac{1}{(km+1)(m-1)+2}} \text{ holds for } \nu > \max(N_0, N_1, N_2, N_3, N_4).$$

Hence we can consider two cases in (34) as follows :

$$\begin{cases} 1) \quad q_\nu \leqslant H(P) < q_{\nu+1}^{\frac{1}{(km+1)(m-1)+2}}, \\ 2) \quad q_{\nu+1}^{\frac{1}{(km+1)(m-1)+2}} \leqslant H(P) < q_{\nu+1}. \end{cases} \tag{35}$$

1) *Suppose that the first relation in* (35) *holds.* If we write the relations (30) and (31) with n replaced by ν we get by using (35) 1) :

$$|P(\gamma)| \geqslant \frac{c_6}{H(P)^{(km+1)(m-1)}} - \frac{c_4}{H(P)^{(km+1)(m-1)+1}},$$

and using (33) :

$$|P(\gamma)| \geqslant \frac{c_6/2}{H(P)^{(km+1)(m-1)}}. \tag{36}$$

2) *If the second relation in* (35) *holds*, writing (30) with n replaced by $\nu + 1$, from (35) 2) we obtain

$$(37) \qquad |P(\gamma)| \geqslant \frac{c_6}{H(P)^{km(m-1)[(km+1)(m-1)+2]+m-1}} - \frac{c_4}{H(P)^{\omega(\nu+1)-1}},$$

and so by using first (35) and then (33) :

$$(38) \qquad |P(\gamma)| \geqslant \frac{c_6/2}{H(P)^{km(m-1)[(km+1)(m-1)+2]+m-2}}.$$

As the exponent of $H(P)$ on the right hand side of (38) is greater than that of (36), (38) is verified for all polynomials $P(x)$ of degree at most $m - 1$ and of height greater than $\max\left(q_{\nu_0}, \frac{2c_4}{c_6}\right)$. This shows us that $\mu(\gamma) \geqslant m$.

This, together with the relation $\mu(\gamma) \leqslant m$ gives us $\mu(\gamma) = m$ also in case $m > 1$.

Note. It follows from the proof of Th. I that, *if ξ is a Liouville number which satisfies the condition*

$$(39) \qquad \liminf_{n \to \infty} w(n) > km(m-1)[(km+1)(m-1)+2] + m + 1,$$

then the conclusion of Th. I is still true.

Special case. *Let α be an algebraic number of degree* m. *If ξ is a Liouville number which satisfies the condition* (39), *then the numbers $\alpha + \xi$ and $\alpha\,\xi$ belong to U_m.*

P. ERDÖS [1] proved that, for every real number r, there exist Liouville numbers $\xi_i (i = 1,2,3,4)$ such that

$$(40) \qquad \mathrm{r} = \xi_1 + \xi_2, \quad \mathrm{r} = \xi_3 \cdot \xi_4.$$

If r is a real algebraic number of degree m ($m > 1$) we have the following

Corollary 1. *Let α be a real algebraic number of degree m $(m > 1)$, and let $\xi_i (i = 1,2,3,4)$ be Liouville numbers which satisfy the relations $\alpha = \xi_1 + \xi_2$, $\alpha = \xi_3 \cdot \xi_4$. Then*

$$\liminf_{n \to \infty} \omega(n)_{\xi_i} \leqslant m^4 - m^3 + m^2 + 1 \qquad (i = 1,2,3,4). \tag{41}$$

Proof. Suppose that $\alpha = \xi_1 + \xi_2$ and $\liminf_{n \to \infty} \omega(n)_{\xi_1} > m^4 - m^3 + m^2 + 1$. If we take $\gamma = \alpha - \xi_1$ in Th. I, we see from (39) for $k = 1$ that

$$\mu(\gamma) = \mu(\alpha - \xi_1) = \mu(\xi_2) \geqslant 2.$$

But this is impossible, since $\mu(\xi_2) = 1$.

Similarly, taking $\gamma = \dfrac{1}{\alpha} \xi_3$ in Th. I, we obtain

$$\mu(\gamma) = \mu\left(\frac{1}{\alpha} \xi_3\right) = \mu(\xi_4^{-1}) \geqslant 2,$$

which is impossible since $\mu(\xi_4^{-1}) = 1$, by a well known property of Liouville numbers.

Hence we have $\liminf_{n \to \infty} \omega(n)_{\xi_i} \leqslant m^4 - m^3 + m^2 + 1$ for $i = 1,2,3,4$.

Corollary 2. *Let ξ be a Liouville number such that $\liminf_{n \to \infty} \omega(n)_\xi >$ $2m(m - 1)[(2m + 1)(m - 1) + 2] + m + 1$. Then, for every natural number k, there are numbers $\gamma_i (i = 1,2,3,4)$ which belong to U_k such that*

$$\xi = \gamma_1 + \gamma_2, \quad \xi = \gamma_3 \cdot \gamma_4. \tag{42}$$

Proof. Let α be a real algebraic number of degree k. We see from Th. I and the property of ξ that, the numbers

$$\gamma_1 = \alpha + \frac{\xi}{2}, \ \gamma_2 = -\alpha + \frac{\xi}{2}, \ \gamma_3 = \alpha \xi^2, \ \gamma_4 = \frac{1}{\alpha \xi}$$

belong to U_k and we have $\xi = \gamma_1 + \gamma_2$, $\xi = \gamma_3 \cdot \gamma_4$.

Theorem II. *Let* $\alpha_i (i = 0, \dots, k)$, $\beta_j (j = 0, \dots, l)$ $(k \geqslant 0,\ l \geqslant 0,\ \max\ (k, l) > 0,\ \alpha_k \neq 0,\ \beta_l = 1)$ *be algebraic numbers, so that* $[Q(\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l) : Q] = m$, *and iet the polynomials* $C(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_k x^k$, $D(x) = \beta_0 + \beta_1 x + \dots + \beta_l x^l$ *be relatively prime. If* ξ *is a strong Liouville number, then the number* $\gamma = \dfrac{C(\xi)}{D(\xi)}$ *belongs to* U_m.

Proof. Let the convergents to the regular continued fraction expansion of the strong Liouville number ξ be $\dfrac{p_n}{q_n}$ $(n = 0,1,\dots)$. Put

$$\left| \xi - \frac{p_n}{q_n} \right| = q_n^{-\omega(n)} . \tag{43}$$

Using Lemma 2, we have

$$\begin{cases} C(\xi) - C\left(\dfrac{p_n}{q_n}\right) = \eta_4 \left(\xi - \dfrac{p_n}{q_n}\right) \cdot C'(\delta_n) \\ D(\xi) - D\left(\dfrac{p_n}{q_n}\right) = \eta_5 \left(\xi - \dfrac{p_n}{q_n}\right) \cdot D'(\widetilde{\delta}_n), \end{cases} \tag{44}$$

where η_4 and η_5 are complex numbers with $0 \leqslant |\eta_4|, |\eta_5| \leqslant 1$ and $\delta_n, \widetilde{\delta}_n$ are real numbers which lie in the interval $\xi \cdots \dfrac{p_n}{q_n}$. Since $\lim\limits_{n \to \infty} \dfrac{p_n}{q_n} = \xi$, and $D(\xi) \neq 0$, there is a natural number N_4 such that, for $n > N_4$ the relations

$$\begin{cases} \left| \dfrac{p_n}{q_n} \right| < 2|\xi|; \quad |\delta_n|, |\widetilde{\delta}_n| < 2|\xi|; \quad \left| C\left(\dfrac{p_n}{q_n}\right) \right| < c_7 \\ |C'(\delta_n)| < c_8; \quad |D'(\widetilde{\delta}_n)| < c_9, \ \dfrac{1}{2}|D(\xi)| < \left| D\left(\dfrac{p_n}{q_n}\right) \right| < c_{10} \end{cases} \tag{45}$$

hold, where c_7, c_8, c_9, c_{10} are positive constants with respect to n.

Now, put $\gamma_n = \dfrac{C\left(\dfrac{p_n}{q_n}\right)}{D\left(\dfrac{p_n}{q_n}\right)}$, and let $P_n(x)$ denote the minimal polynomial of the number γ_n $(n > N_4)$ and let $H(P_n)$ denote the height of $P_n(x)$.

Using Lemma 2 with $P(z) = P_n(z)$; $z_1 = \gamma$, $z_2 = \gamma_n$ $(n > N_4)$ we have

$$P_n(\gamma) - P_n(\gamma_n) = \eta_6(\gamma - \gamma_n)\, P_n'(\tilde{\tilde{\delta}}_n) \qquad (n > N_4), \tag{46}$$

where $\tilde{\tilde{\delta}}_n$ $(n > N_4)$ is a point on the segment $\overline{\gamma\,\gamma_n}$.

Hence there is a real number t with $0 \leqslant t \leqslant 1$, such that

$$\tilde{\tilde{\delta}}_n = (1-t)\frac{C(\xi)}{D(\xi)} + t\,\frac{C\left(\dfrac{p_n}{q_n}\right)}{D\left(\dfrac{p_n}{q_n}\right)} \qquad (n > N_4). \tag{47}$$

Using (45) we get

$$|\tilde{\tilde{\delta}}_n| \leqslant \left|\frac{C(\xi)}{D(\xi)}\right| + \frac{2\,c_7}{|D(\xi)|} = c_{11} \qquad (n > N_4). \tag{48}$$

Let $Q(\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_l) = K$. Since $[K:Q] = m$, the degree of $P_n(x)$ is $\leqslant m$. Using this and (48) we obtain

$$|P_n'(\tilde{\tilde{\delta}}_n)| \leqslant m^2 \cdot \max(1, c_{11}^m) \cdot H(P_n) = c_{12}\, H(P_n) \qquad (n > N_4). \tag{49}$$

On the other hand, we see from (43), (44) and (45) that

$$|\gamma - \gamma_n| = \left|\frac{C(\xi)}{D(\xi)} - \frac{C\left(\dfrac{p_n}{q_n}\right)}{D\left(\dfrac{p_n}{q_n}\right)}\right| \leqslant c_{13}\, q_n^{-\omega(n)}, \tag{50}$$

with a suitable positive constant c_{13}.

Since $P_n(x) = 0$, using (49) and (50) in (46) and putting $c_{12} \cdot c_{13} = c_{14}$ we get

$$(51) \qquad |P_n(\gamma)| \leqslant c_{14}\, q_n^{-\omega(n)}\, H(P_n) \qquad (n > N_4).$$

Now, we shall give an upper bound for the height $h_{\gamma_n} = H(P_n)$ of γ_n $(n > N_4)$. By the definition of γ_n we have

$$\gamma_n\left[\beta_0 + \beta_1\left(\frac{p_n}{q_n}\right) + \dots + \beta_l\left(\frac{p_n}{q_n}\right)^l\right] = \alpha_0 + \alpha_1\left(\frac{p_n}{q_n}\right) + \dots + \alpha_k\left(\frac{p_n}{q_n}\right)^k,$$

so that after multiplying both sides by $q_n^{\max(k,l)}$:

$$(52) \qquad \gamma_n(B_0\,\beta_0 + B_1\,\beta_1 + \dots + B_l\,\beta_l) - (A_0\,\alpha_0 + A_1\,\alpha_1 + \dots + A_k\,\alpha_k) = 0,$$

where $A_i (i = 0,\dots, k)$ and $B_j (j = 0,\dots, l)$ are rational integers with

$$(53) \qquad |A_i|,\ |B_j| \leqslant (\max(1, 2\,|\xi|))^{\max(k,l)} \cdot q_n^{\max(k,l)} = c_{15}\, q_n^{\max(k,l)}$$

$$(i = 0,\dots k;\ \ j = 0,\dots l;\ \ n > N_5),$$

since $\left|\dfrac{p_n}{q_n}\right| < 2\,|\xi|$ for $n > N_4$.

According to this, we can use Lemma 3 with

$$g = m,\ \ d = 1,\ \ l_i = 1\ \ (i = 0, 1 \dots, k + l + 1),\ \ H \leqslant c_{15} \cdot q_n^{\max(k,l)},$$

and we obtain

$$(54) \qquad H(P_n) \leqslant 3^{(k+l+4)m} \cdot q_n^{\max(k,l)\cdot m} \cdot c_{15}^m \prod_{i=0}^{k} (h_{\alpha_i})^m \cdot \prod_{j=0}^{l} (h_{\beta_j})^m$$

or, by putting $3^{(k+l+4)m} \cdot c_{15}^m \prod_{i=0}^{k} (h_{\alpha_i})^m \prod_{j=0}^{l} (h_{\beta_j})^m = c_{16}$:

$$(55) \qquad H(P_n) \leqslant c_{16}\, q_n^{m\cdot\max(k,l)} \qquad (n > N_4).$$

It can be seen easily that, the positive constant c_{16} is not dependent on q_n ; hence there is a natural number N_5 such that $q_n > c_{16}$ for $n > N_5$. Using this, (55) gives

(56) $$H(P_n) < q_n^{m \cdot \max(k,l)+1}$$

for $n > \max(N_4, N_5)$. Using this in (51) we get

(57) $$|P_n(\gamma)| \leqslant c_{14} \frac{H(P_n)}{q_n^{\omega(n)}} \leqslant \frac{c_{14}}{H(P_n)^{\frac{\omega(n)}{m \cdot \max(k,l)+1}-1}} \qquad (n > \max(N_4, N_5)).$$

In the same way as in the proof of the first part of Theorem I, it can be shown that we can extract from $\{P_n(x)\}$ a subsequence $\{P_{n_{j_k}}(x)\}$ such that

(58) $$0 < |P_{n_{j_k}}(\gamma)| \leqslant \frac{c_{14}}{H(P_{n_{j_k}})^{\frac{\omega(n_{j_k})}{m \cdot \max(k,l)+1}-1}}$$

with $\lim\limits_{k\to\infty} H(P_{n_{j_k}}) = +\infty$ and $\lim\limits_{k\to\infty} \omega(n_{j_k}) = +\infty$.

Since the degree of $P_{n_{j_k}}(x) \leqslant m$, the relation (58) shows that

(*) $$\mu(\gamma) \leqslant m.$$

To complete the proof it suffices to show that we have $\mu(\gamma) \geqslant m$. For this we shall distinguish two cases according as $m = 1$ or $m > 1$:

Case 1. If $m = 1$, from the definition of $\mu(\gamma)$ we have $\mu(\gamma) \geqslant 1$ and from above $\mu(\gamma) \leqslant 1$, so that we get $\mu(\gamma) = 1$.

Case 2. Let $m > 1$ and let $P(x)$ be a polynomial of degree f $(0 \leqslant f \leqslant m-1)$ with integral coefficients and let $H(P)$ denote as usual the height of $P(x)$. If we use Lemma 2, we obtain as in the proof of the corresponding part of Theorem I:

(59) $$\begin{cases} P(\gamma) - P(\gamma_n) = \eta_7(\gamma - \gamma_n) \cdot P'(\sigma_n) \\ |P'(\sigma_n)| \leqslant c_{12} H(P), \ |\gamma - \gamma_n| \leqslant c_{13}\, q_n^{-\omega(n)} \end{cases} \qquad (n > N_4),$$

and consequently

(60) $$|P(\gamma) - P(\gamma_n)| \leqslant c_{14}\, H(P) \cdot q_n^{-\omega(n)} \qquad (n > N_4).$$

On the other hand, by Lemma 5, there is a natural number N_6, so that the degree of the algebraic number γ_n is equal to m, for $n > N_6$. Since $f < m$, we have $P(\gamma_n) \neq 0$ for $n > N_6$, so we may apply Lemma 1 with $\alpha = \gamma_n (n > \max(N_4, N_6))$, $s = m$, $n = f$, and we obtain

(61) $$|P(\gamma_n)| \geqslant \frac{1}{(m+1)^f\,(f+1)^{m-1}\,H(P)^{m-1}\,h_{\gamma_n}^f} \quad (n > \max(N_4, N_6)).$$

Using the relation $f \leqslant m-1$ and putting $m^{1-m}(m+1)^{1-m} \cdot c_{16}^{1-m} = c_{17}$, we obtain from (55) and (61) :

(62) $$|P(\gamma_n)| \geqslant \frac{c_{17}}{H(P)^{m-1}\, q_n^{m(m-1)\,.\,\max(k,\,l)}} \qquad (n > \max(N_4, N_6)).$$

(Note that $h_{\gamma_n} = H(P_n)$).

On the other hand, since $P(\gamma_n) \neq 0$ for $n > \max(N_4, N_6)$, we obtain from (60) and (62) that

(63) $$|P(\gamma)| \geqslant \frac{c_{17}}{H(P)^{m-1}\, q_n^{m(m-1)\max(k,\,l)}} - \frac{c_{14}\, H(P)}{q_n^{\omega(n)}} \quad (n > \max(N_4, N_6)).$$

Now, as ξ is taken as a strong Liouville number, there exists a natural number N_7, so that the relation

(64) $$w(n) > m(m-1)\max(k, l)\,[m(m-1)\max(k, l) + m + 1] + m + 1$$

holds for $n > N_7$.

Suppose that the polynomial $P(x)$ satisfies the condition

(65) $$H(P) > \max\left(q_{\nu_0}, \frac{2\,c_{14}}{c_{17}}\right), \;\; \nu_0 > \max(N_4, N_6, N_7).$$

It is clear that, for every polynomial $P(x)$ with (65), there exists a natural number $\nu \geqslant \nu_0$ such that

$$(66) \qquad q_\nu \leqslant H(P) < q_{\nu+1}.$$

Finally, by combining the relations (64) and (31) we obtain the inequality $q_\nu < q_{\nu+1}^{\frac{1}{m(m-1)\max(k,l)+m+1}}$.

Hence, we can consider two cases in (66) as follows :

$$(67) \qquad \begin{cases} \text{a)} \quad q_\nu \leqslant H(P) < q_{\nu+1}^{\frac{1}{m(m-1)\max(k,l)+m+1}}, \\ \text{b)} \quad q_{\nu+1}^{\frac{1}{m(m-1)\max(k,l)+m+1}} \leqslant H(P) < q_{\nu+1}. \end{cases}$$

I — Suppose that the first relation in (67) holds. Writing the relation (63) with n replaced by ν and using (67) a) and (64) we get

$$(68) \qquad |P(\gamma)| \geqslant \frac{c_{17}/2}{H(P)^{m(m-1)\max(k,l)+m-1}} \qquad (H(P) \text{ large}).$$

II — Suppose that the second relation in (67) holds. Writing (63) with n replaced by $\nu + 1$ and using (67) b) and (64) we obtain

$$(69) \qquad |P(\gamma)| \geqslant \frac{c_{17}/2}{H(P)^{m(m-1)\max(k,l)[m(m-1)\max(k,l)+m+1]+m-1}}.$$

Since the degree of the polynomial $P(x)$ can be any natural number f less than m, the relations (68) and (69) show that in any case

$$(\ast\ast) \qquad \mu(\gamma) \geqslant m.$$

From $(\ast)$ and $(\ast\ast)$ we get $\mu(\gamma) = m$ and this completes the proof.

Note. *If we take in Theorem II instead of the strong Liouville number ξ, a Liouville number which satisfies the condition (64), then the Theorem II remains true.*

Now, we shall give a related theorem to Th. I, which is of easier application.

Theorem III. *Let $\alpha_0, \ldots, \alpha_k$ $(k \geqslant 1,\ \alpha_k \neq 0)$ be algebraic numbers and let $[Q(\alpha_0, \ldots, \alpha_k) : Q] = m$, and let ξ be an irrational number which admits a rational approximation sequence $\left\{ \frac{a_i}{b_i} \right\}$ $(a_i,\ b_i \in \mathbf{Z},\ b_i > 1$ for $i > i_0$, with a suitable $i_0)$ satisfying the conditions*

$$1)\quad \lim_{i \to \infty} \frac{\log b_{i+1}}{\log b_i} = +\infty ,$$

$$2)\quad \limsup_{i \to \infty} \frac{\log b_{i+1}}{\log \left| \xi - \frac{a_i}{b_i} \right|^{-1}} < +\infty .$$

Then ξ is a Liouville number and $\gamma = \alpha_0 + \ldots + \alpha_k \xi^k \in U_m$.

Proof. From 1) we have immediately $\lim\limits_{i \to \infty} b_i = +\infty$, and from 1) and 2) we obtain by division

$$\lim_{i \to \infty} \left(\log \left| \xi - \frac{a_i}{b_i} \right|^{-1} / \log b_i \right) = +\infty , \tag{70}$$

which immediately shows that ξ is a Liouville number with $\lim\limits_{i \to \infty} \frac{a_i}{b_i} = \xi$.

In order to prove the second, main assertion of the theorem we shall show first that $\mu(\gamma) \leqslant m$.

If we set

$$\left| \xi - \frac{a_i}{b_i} \right| = b_i^{-\omega_i} , \tag{71}$$

we have by (70)

$$\lim_{i \to \infty} \omega_i = +\infty . \tag{72}$$

Now, let $P_i(x)$ denote the minimal polynomial of the algebraic number

$$\gamma_i = \alpha_0 + \alpha_1 \frac{a_i}{b_i} + \ldots + \alpha_k \left(\frac{a_i}{b_i} \right)^k \qquad (i = 1, 2, \ldots).$$

By a similar reasoning to that given in the corresponding section ((14) - (22)) of the proof of Theorem I we obtain

$$(73) \qquad |P_i(\gamma)| \leqslant \frac{c_{18}}{H(P_i)^{\frac{\omega_i}{km+1}-1}} \qquad \text{for} \quad i > i_1,$$

where i_1 is a suitable natural number, c_{18} is a positive constant which depends only on $k, m, \alpha_0, \dots, \alpha_k$, ξ but not on i and $H(P_i)$ denotes the height of $P_i(x)$.

From (73) we obtain using the fact that ξ is a Liouville number - again an in Theorem I, (22) - (24), - that

$$(74) \qquad 0 < |P_{i_{j_k}}(\gamma)| \leqslant \frac{c_{18}}{H(P_{i_{j_k}})^{\frac{\omega_{i_{j_k}}}{km+1}-1}},$$

with $\lim_{k\to\infty} H(P_{i_{j_k}}) = +\infty$ and $\lim_{k\to\infty} \omega_{i_{j_k}} = +\infty$. The relation (73) shows that

$$(\ast) \qquad \mu(\gamma) \leqslant m.$$

If $m = 1$, we get from $(\ast)$ immediately $\mu(\gamma) = 1$, as we have always $\mu(\gamma) \geqslant 1$.

Next, assume $m > 1$. In this case we shall show that

$$(\ast\ast) \qquad \mu(\gamma) \geqslant m,$$

which together with $(\ast)$ will conclude the proof of the theorem.

Now we can show as in Theorem I ((11) - (20)) that there exist positive constants c_{19} and c_{20} which depend only on $\alpha_j (j = 0, \dots, k)$, k, m, ξ, and a natural number i_2 such that the relations

$$(75) \qquad \left| \frac{a_i}{b_i} \right| < 2|\xi| \qquad (i > i_2),$$

$$(76) \qquad |\gamma - \gamma_i| \leqslant c_{19} \left| \xi - \frac{a_i}{b_i} \right| \qquad (i > i_2),$$

(77) $$H(P_i) \leqslant c_{20}\, b_i^{km} \qquad (i > i_2),$$

hold.

Let $P(x)$ be an arbitrary polynomial of degree f $(0 \leqslant f \leqslant m-1)$ with rational integral coefficients and let $H(P)$ denote the height of $P(x)$. Then, we have by Lemma 2

(78) $$P(\gamma) - P(\gamma_i) = \eta_8(\gamma - \gamma_i) \cdot P'(\rho_i) \qquad (i > i_2),$$

where η_8 is a complex number with $0 \leqslant |\eta_8| \leqslant 1$ and ρ_i is a point on the segment $\overline{\gamma\,\gamma_i}$.

As in the proof of Theorem I ((16) - (17)), there is a positive constant c_{21} depending only on $\alpha_j (j = 0,\ldots, k)$, k, m, ξ such that

(79) $$|P'(\rho_i)| < c_{21}\, H(P) \qquad (i > i_2).$$

Combining the relations (76), (78) and (79), and putting $c_{19} \cdot c_{21} = c_{22}$ we obtain

(80) $$|P(\gamma)| \geqslant |P(\gamma_i)| - c_{22} \cdot \left| \xi - \frac{a_i}{b_i} \right| \cdot H(P) \qquad (i > i_2).$$

Let

(81) $$\lambda = \limsup_{i \to \infty} \frac{\log b_{i+1}}{\log \left| \xi - \frac{a_i}{b_i} \right|^{-1}} .$$

According to the condition 2) of the Theorem, λ is a finite number, which is obviously non - negative.

Let t be a fixed natural number satisfying the inequality

(82) $$t > \lambda .$$

Then

(83) $$t \geqslant 1 ,$$

and by condition 3) we have for sufficiently large i, say for $i > i_3$:

(84)
$$\frac{\log b_{i+1}}{\log \left| \xi - \frac{a_i}{b_i} \right|^{-1}} < t,$$

which is equivalent to

(85)
$$\left| \xi - \frac{a_i}{b_i} \right| < \frac{1}{b_{i+1}^{\frac{1}{t}}} \qquad (i > i_3).$$

(80) and (85) together give us now :

(86)
$$|P(\gamma)| > |P(\gamma_i)| - \frac{c_{22}\, H(P)}{b_{i+1}^{\frac{1}{t}}} \qquad (i > \max(i_2, i_3)).$$

On the other hand by Lemma 5, there exists a natural number i_4, such that for $i > i_4$, γ_i is exactly of degree m. As the degree f of $P(x)$ is at most $m - 1$, we have $P(\gamma_i) \neq 0$ for $i > i_4$. Hence by Lemma 1 we have

(87)
$$|P(\gamma_i)| \geqslant \frac{1}{(f+1)^{m-1}\,(m+1)^f\, H(P)^{m-1}\, H(P_i)^f} \qquad (i > i_4).$$

Using $f \leqslant m - 1$ this gives

(88)
$$|P(\gamma_i)| \geqslant \frac{1}{m^{m-1}\,(m+1)^{m-1}\, H(P)^{m-1}\, H(P_i)^{m-1}} \qquad (i > i_4).$$

(86), (88) and (77) give together

(89)
$$|P(\gamma)| > \frac{c_{23}}{H(P)^{m-1}\, b_i^{km(m-1)}} - \frac{c_{22}\, H(P)}{b_{i+1}^{\frac{1}{t}}} \qquad (i > \max(i_2, i_3, i_4)),$$

where we have put $m^{1-m}(m+1)^{1-m} \cdot c_{20}^{1-m} = c_{23}$.

According to the condition 1) of the Theorem we can find an index i_5, such that the following inequality holds :

(90) $$\log b_{i+1} / \log b_i > \mu \qquad (i > i_5),$$

with $\mu = km(m-1)\,[km(m-1) + m + 1]\,t^2 + (m+1)\,t$.

Finally, suppose that the polynomial $P(x)$ satisfies the further condition

(91) $$H(P) > \max\left(b_{\max(i_2,\, i_3,\, i_4,\, i_5)},\ \frac{2\,c_{22}}{c_{23}}\right) = H_0 .$$

From (90) and (83) we get $b_{i+1} > b_i (i > i_5)$, and it is clear that, for every such polynomial there is a natural number $j \geqslant \max(i_2, i_3, i_4, i_5)$, such that

(92) $$b_j \leqslant H(P) < b_{j+1} .$$

As in the proofs of the two previous theorems we distinguish two cases as follows :

(93) $$\begin{cases} \text{a)} \quad b_j \leqslant H(P) < b_{j+1}^{\frac{1}{t\,[km(m-1)+m+1]}} \\ \text{b)} \quad b_{j+1}^{\frac{1}{t\,[km(m-1)+m+1]}} \leqslant H(P) < b_{j+1} . \end{cases}$$

1 — Suppose that the inequality (93) a) holds. Writing (89) with i replaced by j and using (93) a) and (91) we obtain

(94) $$|P(\gamma)| > \frac{c_{23}/2}{H(P)^{km(m-1)+m-1}} .$$

2 — If the inequality (93) b) holds, we get first by writing (89) with i replaced by $j+1$

(95) $$|P(\gamma)| > \frac{c_{23}}{H(P)^{m-1}\, b_{j+1}^{km(m-1)}} - \frac{c_{22}\, H(P)}{b_{j+2}^{\frac{1}{t}}} .$$

Using the first half of (93) b), (95) becomes

(96) $$|P(\gamma)| > \frac{c_{23}}{H(P)^{t\,[km(m-1)+m+1]+m-1}} - \frac{c_{22}\, H(P)}{b_{j+2}^{\frac{1}{t}}} .$$

Now, (90) with $i = j + 1$ gives

$$b_{j+2}^{\frac{1}{t}} > b_{j+1}^{t\,[km(m-1)+m+1]+m+1}. \tag{97}$$

Using the second half of (93) h) this gives

$$b_{j+2}^{\frac{1}{t}} > H(P)^{t\,[km(m-1)+m+1]+m+1}. \tag{98}$$

Putting (98) in (96) and using (91) gives us at last

$$|P(\gamma)| > \frac{c_{23}/2}{H(P)^{t\,[km(m-1)+m+1]+m-1}}. \tag{99}$$

As the right hand side of (99) is less than that of (94), we have in both cases (93) a) and (93) b) :

$$|P(\gamma)| > \frac{c_{23}/2}{H(P)^{t\,[km(m-1)+m+1]+m-1}}$$

for any polynomial $P(x)$ whose degree $< m$ and whose height $> H_0$.

Therefore $\mu(\xi) \geqslant m$, which concludes the proof of the theorem.

Note. As an example to the Liouville number in Theorem III we can take the number

$$\xi = \frac{1}{2^{0!}} + \frac{1}{2^{1!}} + \frac{1}{2^{2!}} + \dots + \frac{1}{2^{n!}} + \dots .$$

In fact, if we put

$$\frac{a_i}{b_i} = \frac{1}{2^{0!}} + \frac{1}{2^{1!}} + \dots + \frac{1}{2^{i!}} \qquad (i = 0,1,\dots),$$

we have

$$b_i = 2^{i!}, \quad \left| \xi - \frac{a_i}{b_i} \right| < \frac{2}{2^{(i+1)!}} \qquad (i = 0,1,\dots).$$

These relations give us

$$\frac{\log b_{i+1}}{\log b_i} = i + 1\,,$$

$$\frac{\log b_{i+1}}{\log \left| \xi - \frac{a_i}{b_i} \right|^{-1}} < \frac{(i+1)!}{(i+1)! - 1}\,,$$

which show immediately that the conditions 1) and 2) of the Theorem III are satisfied.

CHAPTER II

In this chapter, we shall show directly, i.e. without using the fact $U_m^* = U_m (m = 1,2,\ldots)$, that the classes $U_m^* (m = 1,2,\ldots)$, in the classification of Koksma are not empty.

Koksma's classification. Let ξ he a complex number. Suppose that α is an algebraic number of degree n and $P(x)$ is the irreducible polynomial of α, normalized such that its coefficients are relatively prime and its first coefficient is positive. One then defines the height $H(\alpha)$ of α by $H(\alpha) = H(P)$.

Now put

$$w_n^*(H, \xi) = \min_{\substack{\deg \alpha \leqslant n \\ H(\alpha) \leqslant n \\ \alpha \neq \xi}} |\xi - \alpha|$$

and next put

$$w_n^*(\xi) = \limsup_{H \to \infty} \frac{-\log (H\, w_n^*(H, \xi))}{\log H}\,,$$

$$w^*(\xi) = \limsup_{n \to \infty} \frac{w_n^*(\xi)}{n}\,.$$

$w_n^*(H, \xi)$ is a nonincreasing function of H and the functions $w_n^*(\xi)$ and $w^*(\xi)$ satisfy the respective inequalities $0 \leqslant w_n^*(\xi) \leqslant \infty$,

$0 \leqslant w^*(\xi) \leqslant \infty$. Let $\mu^*(\xi)$ be the smallest number n with $w_n^*(\xi) = \infty$, if such integers exist, otherwise put $\mu^*(\xi) = \infty$.

Call ξ an

A^* — number if $\quad w^*(\xi) = 0\ ,\ \ \mu^*(\xi) = \infty,$

S^* — number if $\quad 0 < w^*(\xi) < \infty,\ \ \mu^*(\xi) = \infty,$

T^* — number if $\quad w^*(\xi) = \infty,\ \ \mu^*(\xi) = \infty,$

U^* — number if $\quad w^*(\xi) = \infty,\ \ \mu^*(\xi) < \infty.$

(See KOKSMA [1]). By the definition of U^*, the set $U_m^* = \{\xi \in U^* \mid \mu(\xi) = m\}$ is a subclass of U^* and $U_m^* \cap U_n^* = \phi$, if $m \neq n$. Hence we have the partition $U^* = \bigcup_{m=1}^{\infty} U_m^*$.

Theorem. *Let* $\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_l (k \geqslant 0, l \geqslant 0, \max(k, l) > 0, \beta_l = 1)$ *be algebraic numbers with* $[Q(\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_l) : Q] = m$ *and let* ξ *be a strong Liouville number. If the polynomials* $C(x) = \alpha_0 + \ldots + \alpha_k x^k$, $D(x) = \beta_0 + \ldots + \beta_l x^l$ *are relatively prime, then* $\gamma = \dfrac{C(\xi)}{D(\xi)}$ *belongs to* U_m^*.

Proof. Let the convergents of the regular continued fraction expansion of ξ be $\dfrac{a_n}{b_n}$ $(n = 1,2,\ldots)$. Put

$$\left| \xi - \frac{a_n}{b_n} \right| = b_n^{-\omega(n)}\ . \tag{1}$$

It is clear that the equation $D(x) = 0$ has only a finite number of solutions in Q, that is, there exist a natural number N_0, such that if $n > N_0$, then $D\left(\dfrac{a_n}{b_n}\right) \neq 0$.

Now we put

(2) $$\gamma_n = \frac{C\left(\frac{a_n}{b_n}\right)}{D\left(\frac{a_n}{b_n}\right)} \qquad (n > N_0).$$

By the definition of the algebraic number γ_n, the value of the polynomial

(3) $$F_n(y, x_0, \dots x_k, \dots, x_{k+l+1}) = b_n^{\max(k,l)}\, y\left(x_{k+1} + \left(\frac{a_n}{b_n}\right) x_{k+2} + \dots + \left(\frac{a_n}{b_n}\right)^l x_{k+l+1}\right) - b_n^{\max(k,l)} \cdot \left(x_0 + \frac{a_n}{b_n} x_1 + \dots + \left(\frac{a_n}{b_n}\right)^k x_k\right)$$

is zero for $y = \gamma_n$, $x_i = \alpha_i (i = 0, \dots, k)$, $x_{k+j+1} = \beta_j (j = 0, \dots, l)$.

On the other hand, since $\lim_{n \to \infty} \frac{a_n}{b_n} = \xi$, $\xi \neq 0$, there is a natural number N_1, such that if $n > N_1$ then $|a_n| < 2 |\xi| b_n$.

Hence we have

(4) $$H_n \leqslant (\max(1, c_1))^{\max(k,l)} \cdot b_n^{\max(k,l)} \qquad (n > \max(N_0, N_1)),$$

where $c_1 = 2|\xi|$ and H_n is the maximum of the absolute values of the coefficients of $F_n(y, x_0, \dots, x_{k+l+1})$.

Now, by Lemma 3 in Chapter I and by (4) we obtain

(5) $$H_{\gamma_n} \leqslant c_2\, b_n^{\max(k,l) \cdot m} \quad \text{for} \quad n > \max(N_0, N_1),$$

where c_2 is a positive constant, which depends on $\xi, m, k, l, \alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l$, but not on H_{γ_n}.

As $b_n \to +\infty$ for $n \to \infty$, we have $c_2 \leqslant b_n$ for $n > N_2$, and we obtain from (5) :

(6) $$H_{\gamma_n} \leqslant b_n^{m \cdot \max(k,l)+1}.$$

Next, by using Lemma 2 in Chapter I, we get

$$(7)\qquad \begin{cases} C(\xi) = C\left(\dfrac{a_n}{b_n}\right) + \left(\xi - \dfrac{a_n}{b_n}\right)\cdot t_1(n), & |t_1(n)| < c_3, \\ D(\xi) = D\left(\dfrac{a_n}{b_n}\right) + \left(\xi - \dfrac{a_n}{b_n}\right)\cdot t_2(n), & |t_2(n)| < c_4, \end{cases}$$

where c_3 and c_4 are positive constants. Hence from (1) and (7) we have

$$(8)\qquad |\gamma - \gamma_n| \leqslant \frac{\left|D\left(\dfrac{a_n}{b_n}\right)\right|\cdot|t_1(n)| + \left|C\left(\dfrac{a_n}{b_n}\right)\right|\cdot|t_2(n)|}{|D(\xi)|\cdot\left|D\left(\dfrac{a_n}{b_n}\right)\right|}\cdot b_n^{-\omega(n)}.$$

Since $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \xi$, there is a natural number N_3 and a positive constant c_5, so that the relations

$$(9)\qquad \left|D\left(\frac{a_n}{b_n}\right)\right|,\ \left|C\left(\frac{a_n}{b_n}\right)\right| < c_5,\quad \left|D\left(\frac{a_n}{b_n}\right)\right| > \frac{1}{2}|D(\xi)| > 0$$

hold for $n > N_3$. Combining the relations (8) and (9) we obtain

$$(10)\qquad 0 < |\gamma - \gamma_n| \leqslant c_6\, b_n^{-\omega(n)} \qquad (n > \max(N_0, N_1, N_2, N_3) = \bar{N}),$$

where c_6 is again a positive constant. ($\gamma = \gamma_n$ is impossible, as this would entail that ξ is algebraic.)

As $\limsup\limits_{n\to\infty}\omega(n) = +\infty$, we can choose a subsequence $\{\omega(n_j)\}$, such that $\lim\limits_{j\to\infty}\omega(n_j) = +\infty$. As b_{n_j} tend to $+\infty$ with $j\to\infty$, (10) with $n = n_j$ ($j = 1,2,\ldots$) gives us, that $\{\gamma_{n_j}\}$ has an infinite number of different terms (otherwise $b_{n_j}^{-\omega(n_j)}$ would have a positive lower bound). If we put $H_{\gamma_n} = H(\gamma_n)$, the sequence $\{H(\gamma_{n_j})\}$ has a subsequence $\{H(\gamma_{n_{j_k}})\}$ tending to $+\infty$

(otherwise the sequence $\{\gamma_{n_j}\}$ as consisting of algebraic numbers of bounded height and bounded degree would contain only a finite number of different terms).

Finally putting (6) in (10) we get for $\{\gamma_{n_{j_k}}\}$

$$(11) \qquad 0 < |\gamma - \gamma_{n_{j_k}}| \leqslant \frac{c_6}{H(\gamma_{n_{j_k}})^{\frac{\omega(n_{j_k})}{m \cdot \max(k, l)+1}}} \qquad \text{for } n_{j_k} > \bar{N}.$$

(11) gives us $\mu^*(\gamma) \leqslant m$. To prove the opposite inequalty $\mu^*(\gamma) \geqslant m$ we distinguish two cases as follows :

I — If $m = 1$, then $\mu^*(\gamma) \leqslant 1$ and as always $\mu^*(\gamma) \geqslant 1$, so $\mu^*(\gamma) = 1$. Hence in this case the proof is complete.

II — Suppose that $m > 1$. Let β be an algebraic number of degree $s(1 \leqslant s \leqslant m - 1)$ and let $H(\beta)$ be the height of β. By Lemma 5 in Chapler I, there exists a natural number N_4, such that the degree of the algebraic number γ_n is equal to m, if $n > N_4$. On the other hand, since $s \leqslant m - 1$, the minimal polynomial of β is different from the minimal polynomial of $\gamma_n (n > N_4)$. Hence we may use Lemma 4 in Chapter I with (5), and we obtain

$$(12) \quad |\gamma_n - \beta| \geqslant \frac{1}{2^{m-1} m^m (m+1)^{m-1} (\max(1, c_2))^{m-1} H(\beta)^m b_n^{m(m-1) \max(k, l)}}$$

and putting $2^{1-m} m^{-m} (m+1)^{1-m} (\max(1, c_2))^{1-m} = c_7$ we have

$$(13) \qquad |\gamma_n - \beta| \geqslant \frac{c_7}{H(\beta)^m b_n^{m(m-1) \max(k, l)}} \qquad (n > \max(N_0, N_1, N_2, N_3, N_4)).$$

Next, using the inequality $|\gamma - \beta| = |(\gamma_n - \beta) + (\gamma - \gamma_n)| \geqslant |\gamma_n - \beta| - |\gamma - \gamma_n|$, and (10), (13) we obtain

$$(14) \qquad |\gamma - \beta| \geqslant \frac{c_7}{H(\beta)^m b_n^{m(m-1) \max(k, l)}} - \frac{c_6}{b_n^{\omega(n)}}.$$

Now, since ξ is strong, then there is a natural number N_5 such that the inequality

$$(15) \qquad \omega(n) > m(m-1)\max(k,l)\,[m(m-1)\max(k,l)+m+1]+m+1$$

holds for $n > N_5$. Finally, suppose that the algebraic number β satisfies the condition

$$(16) \qquad H(\beta) > \max\left(b_{\max(N_0, N_1, N_2, N_3, N_4, N_5)}, \frac{2c_6}{c_7}\right) = H_0 .$$

It is clear that, for every $H(\beta)$ with (16), there exists a natural number j, such that

$$(17) \qquad b_j \leqslant H(\beta) < b_{j+1} .$$

On the other hand, since $b_{j+1}^{\frac{1}{m(m-1)\max(k,l)+m+1}} \geqslant b_j$, we can consider two cases in (17) as follows :

$$(18) \qquad \begin{cases} \text{a)} \quad b_j \leqslant H(\beta) < b_{j+1}^{\frac{1}{m(m-1)\max(k,l)+m+1}} \\ \text{b)} \quad b_{j+1}^{\frac{1}{m(m-1)\max(k,l)+m+1}} \leqslant H(\beta) < b_{j+1} . \end{cases}$$

I — Suppose that (18) a) holds. Then writing (14) with n replaced by j and using (15), (16) and (18) a), we obtain

$$(19) \qquad |\gamma - \beta| \geqslant \frac{c_7}{H(\beta)^{m(m-1)\max(k,l)+m}} - \frac{c_6}{H(\beta)^{m(m-1)\max(k,l)+m+1}}$$

$$\geqslant \frac{c_7/2}{H(\beta)^{m(m-1)\max(k,l)+m}}$$

for $H(\beta) > H_0$.

II — If (18) b) holds, writing (14) with n replaced by $j+1$ and using (15), (16) and (18) b) we have

$$|\gamma - \beta| \geqslant \frac{c_7/2}{H(\beta)^{m(m-1)\max(k,l)\,[m(m-1)\max(k,l)+m+1]+m}} \tag{20}$$

for $H(\beta) > H_0$.

Hence the relations (19) and (20) show that $\mu^*(\gamma) \geqslant m$. But we had $\mu^*(\gamma) \leqslant m$, therefore $\mu^*(\gamma) = m$, and the proof is completed.

CHAPTER III

In this chapter, we shall show that the classes $U_m\,(m = 1,2,\ldots)$ for the Hensel's field Q_p of p-adic numbers are not empty.

Mahler's classification in Q_p. Let $P(x)$ be a polynomial with integral coefficients and $H(P)$ be the height of $P(x)$.

Suppose that m and A are two natural number and $\alpha \in Q_p$.

Then Mahler puts

$$\omega_m(\alpha \mid A) = \min_{\substack{\deg P \leqslant m \\ H(P) \leqslant A \\ P(\alpha) \neq 0}} (|P(\alpha)|_p).$$

It is clear that $0 \leqslant \omega_m(\alpha \mid A) \leqslant 1$, since, if $P(x) = 1$, then $|P(\alpha)|_p = 1$.

Next Mahler puts

$$\omega_m(\alpha) = \limsup_{A \to \infty} \frac{-\log \omega_m(\alpha \mid A)}{\log A}$$

and

$$\omega(\alpha) = \limsup_{m \to \infty} \frac{\omega_m(\alpha)}{m}.$$

By what we said above, $\omega_m(\alpha)$ as a function of m is nondecreasing. One has, $0 \leqslant \omega_m(\alpha) \leqslant \infty$ and $0 \leqslant \omega(\alpha) \leqslant \infty$.

If $\omega_m(\alpha) = \infty$ for some integer m, let $\mu(\alpha)$ be the smallest such integer; if $\omega_m(\alpha) < \infty$ for every m, put $\mu(\alpha) = \infty$.

Mahler calls the number α an

A — number if $\omega(\alpha) = 0$, $\mu(\alpha) = \infty$,

S — number if $0 < \omega(\alpha) < \infty$, $\mu(\alpha) = \infty$,

T — number if $\omega(\alpha) = \infty$, $\mu(\alpha) = \infty$,

U — number if $\omega(\alpha) = \infty$, $\mu(\alpha) < \infty$

(K. MAHLER [2]). By the definition of U, the set $U_m = \{\alpha \in U \mid \mu(\alpha) = m\}$ is a subset of U and we have $U = \bigcup_{m=1}^{\infty} U_m$.

It is clear that, U_1 is not empty; for example the p - adic number $\sum_{n=1}^{\infty} p^{n!}$ belongs to U_1. Now, to prove that U_m is not empty, we shall use following lemmas :

Lemma 1. *Let $P(x) = a_0 + a_1 x + \dots + a_{m_0} x^{m_0}$ be a polynomial of degree m_0 with integral coefficients and α be a p - adic algebraic number of degree M with $P(\alpha) \neq 0$. Then the relation*

$$|P(\alpha)|_p \geqslant \frac{p^{(M-1)t}}{(M + m_0)!\, H(P)^M\, H(\alpha)^{m_0}}$$

holds, where $|\alpha|_p = p^{-h}$, $t = \min(0, h)$, and $H(P)$, $H(\alpha)$ are the height of $P(x)$ and the height of the minimal polynomial of the algebraic number α respectively (K. MAHLER [2], *P.* 179 - 181).

Lemma 2. *Let $\alpha_0, \dots, \alpha_k$, $\beta_0, \dots, \beta_l$ ($k \geqslant 0$, $l \geqslant 0$, $\max(k, l) \geqslant 1$, $\alpha_k \neq 0$, $\beta_l = 1$) be algebraic number in Q_p. If the polynomials $C(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_k x^k$, $D(x) = \beta_0 + \beta_1 x + \dots + \beta_l x^l$ are relatively prime, then for $x \in Q_p$ the p - adic number $\frac{C(x)}{D(x)}$ is a primitive element* *of the field* $Q(\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l) = K$ *except for only a finite number of values of* x.

Lemma 3. *Let* $\alpha_1, \dots, \alpha_k$ $(k \geqslant 1)$ *be algebraic numbers in* Q_p *with* $[Q(\alpha_1, \dots, \alpha_k) : Q] = g$ *and let* $F(y, x_1, \dots, x_k)$ *be a polynomial with integral coefficients, whose degree in* y *is at least one. If* η *is an algebraic number such that* $F(\eta, \alpha_1, \dots, \alpha_k) = 0$, *then the degree of* $\eta \leqslant dg$ *and*

$$h_\eta \leqslant 3^{2dg + (l_1 + \dots l_k)g} \cdot H^g \, h_{\alpha_1}^{l_1 g} \cdots h_{\alpha_k}^{l_k g},$$

where h_η *is the height of* η, h_{α_i} *is the height of* $\alpha_i (i = 1, \dots, k)$, H *is the maximum of the absolute values of the coefficients of* F, l_i *is the degree of* F *in* $x_i (i = 1, \dots, k)$, *and* d *is the degree of* F *in* y.

The proof is the same as in the Lemma 3 in Chapter I.

Theorem I. *Let* $\alpha_0, \dots, \alpha_k$, $\beta_0, \dots, \beta_l$ $(k \geqslant 0,\ l \geqslant 0,\ \max(k, l) > 0,$ $\alpha_k \neq 0,\ \beta_l = 1)$ *be algebraic numbers in* Q_p *with* $[Q(\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l) : Q] = m$, *and* $\xi \in Q_p$ *be a* p-*adic number, whose canonical form is*

$$\xi = a_0 p^{u_0} + a_1 p^{u_1} + \dots + a_n p^{u_n} + \dots \quad (0 < a_n < p,\ a_n \in \mathbf{Z} \quad (n = 0, 1, \dots),$$

where $u_0 \geqslant 0$, $\lim\limits_{n \to \infty} \dfrac{u_{n+1}}{u_n} = \infty$.

If the polynomials $C(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_k x^k$, $D(x) = \beta_0 + \beta_1 x + \dots + \beta_l x^l$ *are relatively prime, then the* p-*adic number* $\gamma = \dfrac{C(\xi)}{D(\xi)}$ *belongs to* U_m.

Proof. Let us put

$$(1) \quad \xi_n = a_0 p^{u_0} + a_1 p^{u_1} + \dots + a_n p^{u_n}, \ \rho_n = a_{n+1} p^{u_{n+1}} + \dots \ (n = 0, 1, \dots)).$$

By approximating ξ with ξ_n and taking into account the condition $\lim\limits_{n \to \infty} \dfrac{u_{n+1}}{u_n} = +\infty$, we see easily that $\xi \in U_1$.

We have

$$\xi = \xi_n + \rho_n \qquad (n = 0,1,\ldots), \tag{2}$$

and so

$$\begin{cases} C(\xi) = C(\xi_n) + \rho_n [\alpha_1 + \alpha_2(2\,\xi_n + \rho_n) + \ldots + \alpha_k(k\,\xi_n^{k-1} + \ldots + \rho_n^{k-1})] \\ D(\xi) = D(\xi_n) + \rho_n [\beta_1 + \beta_2(2\,\xi_n + \rho_n) + \ldots + \beta_l(l\,\xi_n^{l-1} + \ldots + \rho_n^{l-1})]. \end{cases} \tag{3}$$

Next put

$$\begin{cases} \alpha_1 + \alpha_2(2\,\xi_n + \rho_n) + \ldots + \alpha_k(k\,\xi_n^{k-1} + \ldots + \rho_n^{k-1}) = \tilde{\delta}_n \\ \beta_1 + \beta_2(2\,\xi_n + \rho_n) + \ldots + \beta_l(l\,\xi_n^{l-1} + \ldots + \rho_n^{l-1}) = \tilde{\tilde{\delta}}_n. \end{cases} \quad (n = 0,1,\ldots). \tag{4}$$

It is clear that the equation $D(x) = 0$ has only finitely many solutions in Q, hence there exists a natural number N_0, such that $D(\xi_n) \neq 0$ for every $n > N_0$. Hence by the definition of γ and by (3) we obtain

$$\gamma = \frac{C(\xi)}{D(\xi)} = \frac{C(\xi_n)}{D(\xi_n)} + \rho_n \frac{D(\xi_n)\,\tilde{\delta}_n - C(\xi_n)\,\tilde{\tilde{\delta}}_n}{D(\xi_n)\;D(\xi)} \qquad (n > N_0), \tag{5}$$

and so putting

$$\gamma_n = \frac{C(\xi_n)}{D(\xi_n)}, \quad \sigma_n = \frac{D(\xi_n)\,\tilde{\delta}_n - C(\xi_n)\,\tilde{\tilde{\delta}}_n}{D(\xi_n)\;D(\xi)} \qquad (n > N_0),$$

we have

$$\gamma = \gamma_n + \rho_n\,\sigma_n \qquad (n > N_0). \tag{6}$$

Let $|\alpha_i|_p = p^{-h_i}\,(i = 0,1,\ldots k)$, $|\beta_j|_p = p^{-e_j}\,(j = 0,1,\ldots l)$,

$t_0 = \min(0, h_0, \ldots h_k)$, $t_1 = \min(0, e_0, e_1 \ldots e_l)$, $t_2 = \max(0, e_0, \ldots e_l)$.

Now, since $u_0 \geqslant 0$, ξ and ξ_n are p-adic integers. Hence, by definitions of the p-adic numbers $\tilde{\delta}_n$, $\tilde{\tilde{\delta}}_n$, γ_n, σ_n $(n > N_0)$, we see that

$$
(7) \qquad \begin{cases} |\widetilde{\delta}_n|_p \leqslant p^{-t_0}, & |\widetilde{\widetilde{\delta}}_n|_p \leqslant p^{-t_1} \\ |\gamma_n|_p \leqslant p^{t_2-t_0}, & |\sigma_n|_p \leqslant p^{2t_2-t_0-t_1} \end{cases} \qquad (n > N_0).
$$

Now, let

$$
(8) \qquad P_n(x) = b_0^{(n)} + b_1^{(n)} x + \dots + b_f^{(n)} x^f \qquad (f \leqslant m,\ n > N_0)
$$

be the minimal polynomial of $\gamma_n (n > N_0)$ and $H(P_n)$ be the height of $P_n(x)$. We see from (6) that

$$
(9) \qquad P_n(\gamma) = P_n(\gamma_n + \rho_n \sigma_n) \qquad (n > N_0),
$$

and so

$$
(10) \qquad P_n(\gamma) = P_n(\gamma_n) + \rho_n [b_1^{(n)} \sigma_n + \dots + b_f^{(n)}(f \gamma_n^{f-1} \sigma_n + \dots + \rho_n^{f-1} \sigma_n^f)]
$$

or, putting $b_1^{(n)} \sigma_n + \dots + b_f^{(n)}(f \gamma_n^{f-1} \sigma_n + \dots + \rho_n^{f-1} \sigma_n^f) = \widetilde{\sigma}_n$, we have

$$
(11) \qquad P_n(\gamma) = P_n(\gamma_n) + \rho_n \widetilde{\sigma}_n \qquad (n > N_0).
$$

But we have $P_n(\gamma_n) = 0$, hence using this and (1), (11) and (7) we obtain that

$$
(12) \qquad |P_n(\gamma)|_p \leqslant \frac{p^{m(2t_2-t_0-t_1)}}{p^{u_{n+1}}} = \frac{c_1}{p^{u_{n+1}}} \qquad (n > N_0).
$$

It is clear that c_1 is a positive constant.

Now, we shall give an upper bound for $H(P_n)$ (n $> N_0$). Since $\gamma_n (\beta_0 + \beta_1 \xi_n + \dots + \beta_l \xi_n^l) - (\alpha_0 + \alpha_1 \xi_n + \dots + \alpha_k \xi_n^k) = 0$, the value of the function

$$
F(y, x_0, \dots, x_{k+l+1}) = y(x_{k+1} + \xi_n x_{k+2} + \dots + \xi_n^l x_{k+l+1}) - x_0 - \xi_n x_1 - \dots - \xi_n^k x_k
$$

is zero for $y=\gamma_n (n>N_0)$, $x_i=\alpha_i (i=0,\ldots,k)$, $x_{k+j+1}=\beta_j (j=0,\ldots,l)$ and the maximum of the absolute values of the coefficients of $F(y, x_0, \ldots, x_{k+l+1})$ is at most $p^{2\max(k,l)\cdot u_n}$ $(n>N_0)$.

Using this in Lemma 3, we have

$$h_{\gamma_n} = H(P_n) \leqslant 3^{(k+l+4)m} \cdot p^{2m\max(k,l)u_n} \cdot h_{\alpha_0}^m \cdots h_{\alpha_k}^m \cdot h_{\beta_0}^m \cdots h_{\beta_l}^m , \tag{13}$$

or putting $c_2 = 3^{(k+l+4)m} \cdot h_{\alpha_0}^m \cdots h_{\alpha_k}^m \cdot h_{\beta_0}^m \cdots h_{\beta_l}^m$,

$$H(P_n) \leqslant c_2 \cdot p^{2\max(k,l)u_n} \qquad (n>N_0). \tag{14}$$

Here, since c_2 is a constant and $u_n \to \infty$ for $n \to \infty$, there exists a natural number N_1, such that

$$c_2 < p^{u_n} \qquad \text{for} \quad n > N_1 . \tag{15}$$

Hence from the relations (12) and (15) we obtain that

$$|P(\gamma_n)|_p \leqslant \frac{c_1}{p^{u_{n+1}}} \leqslant \frac{c_1}{\left(H(P_n)\right)^{\frac{u_{n+1}}{[2\max(k,l)\cdot m+1]u_n}}} \qquad (n>\max(N_0, N_1)). \tag{16}$$

Let us put $\dfrac{u_{n+1}}{u_n} = s_n$, so that (16) can be written as

$$|P_n(\gamma)|_p \leqslant \frac{c_1}{H(P_n)^{\frac{s_n}{2\max(k,l)m+1}}} , \tag{17}$$

where $s_n \to \infty$.

By a reasoning exactly similar to that used in the proof of Theorem I of Chapter I (from (22) to (24)), we conclude from (17) that $\mu(\gamma) \leqslant m$.

To complete the proof we have now to prove the opposite inequality $\mu(\gamma) \geqslant m$. To this end we distinguish two cases according as $m=1$ or $m>1$:

1 — Let $m = 1$. Then we have $\mu(\gamma) = 1$ as in the proof of Theorem I, Chapter I and the proof is complete for this case.

2 — Suppose that $m > 1$. Let $P(x) = A_0 + A_1 x + \dots + A_s x^s$ ($A_s \neq 0$, $s \leqslant m - 1$) be a polynomial with integral coefficients and $H(P)$ be the height of $P(x)$. As in (10), we have by (6)

$$P(\gamma) = P(\gamma_n) + \rho_n [A_1 \sigma_n + \dots + A_s(s\, \gamma_n^{s-1} \sigma_n + \dots + \rho_n^{s-1} \sigma_n^s)], \tag{18}$$

or putting

$$A_1 \sigma_n + \dots + A_s(s\, \gamma_n^{s-1} \sigma_n + \dots + \rho_n^{s-1} \sigma_n^s) = \approx{\sigma}_n ,$$

we obtain that

$$P(\gamma) = P(\gamma_n) + \rho_n \approx{\sigma}_n \qquad (n > N_0), \tag{19}$$

and we see from the definition of $\approx{\sigma}_n$ and (7) that

$$|\approx{\sigma}_n|_p \leqslant p^{m(2t_2 - t_0 - t_1)} \qquad (n > N_0). \tag{20}$$

On the other hand, by Lemma 2 there exists a natural number N_2, such that if $n > N_2$, then the degree of γ_n is equal to m. Thus $P(\gamma_n) \neq 0$ for $n > N_2$, and we may use Lemma 1 with $|\gamma_n|_p = p^{t_2 - t_0}$, $M = m, m_0 = s$, and we obtain

$$|P(\gamma_n)|_p \geqslant \frac{p^{-s(t_2 - t_0)}}{(m + s)!\, H(P)^m\, H(P_n)^s} \qquad (n > \max N_0, N_2), \tag{21}$$

and so using (14) in (21) and putting $c_3 = \dfrac{p^{-m(t_2 - t_0)}}{(2m - 1)!\, c_2^{m-1}}$ we have

$$|P(\gamma_n)|_p \geqslant \frac{c_3}{H(P)^m\, p^{2m(m-1)\max(k,l)\, u_n}} \qquad (n > \max(N_0, N_2)). \tag{22}$$

Now by the assumption $\lim\limits_{n \to \infty} \dfrac{u_{n+1}}{u_n} = +\infty$, there exists a natural number N_3, such that if $n > N_3$, then the relation

$$\text{(23)} \qquad \frac{u_{n+1}}{u_n} > 2m(m-1)\max(k,l)\,[2m(m-1)\max(k,l)+m+1]+m+1$$

holds. Next suppose that, $H(P)$ satisfies the condition

$$\text{(24)} \qquad H(P) > \max\left(p^{u_{\max(N_0,N_2,N_3)}}, \frac{c_1}{c_3}\right) = H_0 .$$

For every $H(P)$ with (24) there exists a natural number j, such that

$$\text{(25)} \qquad p^{u_j} \leqslant H(P) < p^{u_{j+1}} .$$

Now, from (23), we have two cases in (25) as follows :

$$\text{(26)} \qquad \begin{cases} \text{a)} \quad p^{u_j} \leqslant H(P) < p^{\frac{u_{j+1}}{2m(m-1)\max(k,l)+m+1}} , \\ \text{b)} \quad p^{\frac{u_{j+1}}{2m(m-1)\max(k,l)+m+1}} \leqslant H(P) < p^{u_{j+1}} . \end{cases}$$

I — If the case (26) a) holds, writing (1), (20), (22) with n replaced by j, we obtain

$$\text{(27)} \qquad |P(\gamma_j)|_p \geqslant \frac{c_3}{H(P)^{2m(m-1)\max(k,l)+m}} ,$$

$$\text{(28)} \qquad |\rho_j \overset{\approx}{\sigma}_j|_p \leqslant \frac{c_1}{p^{u_{j+1}}} \leqslant \frac{c_1}{H(P)^{2m(m-1)\max(k,l)+m+1}} .$$

Next, writing (16) and (19) with n replaced by j and combining the relations (23), (25), (27), (28) and (as a consequence of (27) and (28))

$$\text{(29)} \qquad |P(\gamma)|_p = \max\left(|P(\gamma_j)|_p , |\rho_j \overset{\approx}{\sigma}_j|_p\right) = |P(\gamma_j)|_p$$

we see that

$$\text{(30)} \qquad |P(\gamma)|_p \geqslant \frac{c_3}{H(P)^{2m(m-1)\max(k,l)+m}} \qquad \text{for} \quad H(P) > H_0 .$$

II — Suppose that (26) b) holds. If we write (1), (20), (22) with n replaced by $j+1$, then we have

$$|P(\gamma_{j+1})|_p \geqslant \frac{c_3}{H(P)^{2m(m-1)\max(k,l)[2m(m-1)\max(k,l)+m+1]+m}}, \tag{31}$$

$$|\rho_{j+1}\,\overset{\approx}{\sigma}_{j+1}|_p \leqslant \frac{c_3}{H(P)^{2m(m-1)\max(k,l)[2m(m-1)\max(k,l)+m+1]+m+1}}. \tag{32}$$

But it follows from (31) and (32) that

$$|P(\gamma)|_p = \max\left(|P(\gamma_{j+1})|_p,\ |\rho_{j+1}\,\overset{\approx}{\sigma}_{j+1}|_p\right) = |P(\gamma_{j+1})|_p,$$

and so we obtain

$$|P(\gamma)|_p \geqslant \frac{c_3}{H(P)^{2m(m-1)\max(k,l)[2m(m-1)\max(k,l)+m+1]+m}}. \tag{33}$$

The relations (30) and (33) show that, if $P(x)$ is a polynomial of degree $f\,(f \leqslant m-1)$ with integral coefficients and $H(P)$ is sufficiently large, then

$$|P(\gamma)|_p \geqslant c_3 \cdot H(P)^{-2m(m-1)\max(k,l)[2m(m-1)\max(k,l)+m+1]-m}. \tag{34}$$

By the definition of $\mu(\gamma)$, (34) gives $\mu(\gamma) \geqslant m$ and thus we have $\mu(\gamma) = m$, and the proof is completed for $m > 1$.

Special case. *Let α be a p-adic algebraic number of degree m, and ξ be a p-adic number verifying the conditions of Theorem I. Then $\alpha + \xi$, $\alpha \cdot \xi \in U_m$.*

It can be easily seen from the proof of Theorem I, that it is sufficient to suppose $\limsup_{n\to\infty} \frac{u_{n+1}}{u_n} = +\infty$ and the condition (23), instead of the stronger assumption $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = +\infty$. Hence we have the following :

Corollary. *If the p-adic number ξ in Theorem I has the canonical from $\xi = a_0 p^{u_0} + a_1 p^{u_1} + \ldots + a_n p^{u_n} + \ldots$, $u_0 \geqslant 0$ and such that $\limsup_{n\to\infty} \frac{u_{n+1}}{u_n} = +\infty$*

and $\liminf_{n\to\infty} \frac{u_{n+1}}{u_n} > 2m(m-1)\max(k,l)\,[2m(m-1)\max(k,l)+m]+m+1,$
then Theorem I holds also in this more general case.

Theorem II. *Let* $\alpha_0,\ldots,\alpha_k\ (k \geqslant 1,\ \alpha_k \neq 0)$ *be* p - *adic algebraic numbers in* Q_p *with* $[Q(\alpha_0,\ldots,\alpha_k):Q]=m$, *and* ξ *be a* p - *adic number in the canonical from*

$$\xi = a_0 p^{u_0} + a_1 p^{u_1} + \ldots + a_\nu p^{u_\nu} + \ldots$$

$$(u_0 \geqslant 0,\ u_{\nu+1} > u_\nu,\ a_\nu \in \mathrm{N},\ 0 < a_\nu \leqslant p-1\ \ (\nu = 0,1,\ldots)).$$

Further suppose that the sequence $\{u_\nu\}$ *has a subsequence* $\{u_{\nu_n}\}$ *verifying the conditions*

1) $$\lim_{n\to\infty} \frac{u_{\nu_{n+1}}}{u_{\nu_n}} = +\infty,$$

2) $$\limsup_{n\to\infty} \frac{u_{\nu_{n+1}}}{u_{\nu_n+1}} < +\infty.$$

Then the p - *adic number* $\gamma = \alpha_0 + \alpha_1\xi + \ldots + \alpha_k \xi^k$ *belongs to the* p - *adic* U_m *class.*

We approximate ξ by $\xi_{\nu_n} = a_0 p^{u_0} + \ldots + a_{\nu_n} p^{u_{\nu_n}}$. From 1) and 2) we see easily that ξ is a p - adic U_1 (Liouville) number. The proof, which we shall omit, can be conducted by using a combination of the arguments used in the proofs of the Theorem I above (adapted to the special case $D(x) = 1$) and the Theorem III of Chapter I.

We conclude with some examples :

1) As an example for a p - adic number ξ verifying the conditions of Theorem I above we can take

$$\xi_1 = 1 + p^{1!} + p^{2!} + \ldots + p^{n!} + \ldots,$$

which can be seen at once.

2) As an example for a number ξ of Theorem II above we can take

$$\xi_2 = 1 + p^{1!} + (p^{2!} + p^{2!+1} + p^{2!+2}) + \dots$$
$$+ (p^{n!} + p^{n!+1} + \dots + p^{n!+n}) + \dots .$$

For ξ_2, if we define

$$u_{\nu_0} = 0\,,\ u_{\nu_1} = 1!,\ u_{\nu_n} = n! + n \quad (n \geqslant 2),$$

we see that $u_{\nu_n + 1} = (n+1)!$, and consequently

$$\lim_{n\to\infty} \frac{u_{\nu_{n+1}}}{u_{\nu_n}} = \lim_{n\to\infty} \frac{(n+1)! + (n+1)}{n! + n} = +\infty,$$

$$\limsup_{n\to\infty} \frac{u_{\nu_{n+1}}}{u_{\nu_n + 1}} = \lim_{n\to\infty} \frac{(n+1)! + (n+1)}{(n+1)!} = 1,$$

so that all the conditions on ξ are verified.

REFERENCES

[1] L. BIEBERBACH : Lehrbuch der Funktionentheorie, Baad I. Berlin - Leipzig (1934).

[2] P. ERDÖS : *Representations of real numbers as sums and products of Liouville numbers.* Michigan Math. J. **9** (1962), 59 - 60.

[3] R. GÜTING : *Approximation of algebraic numbers by algebraic numbers.* Michigan Math. J. **8** (1961), 149 - 159.

[4] O.Ş. İÇEN : *Anhang zu den Arbeiten "Über die Funktionswerte der p-adisch elliptischen Funktionen I und II".* Revue de la Fac. de Sci. de l'Université d'Istanbul, Ser. A. **8** (1973), 25 - 35.

[5] J.F. KOKSMA : *Über die Mahlersche Klasseneinteilung der transzendenten Zahlen und die Approximation komplexer Zahlen durch algebraische Zahlen.* Monatshefte Math. Physik **48** (1939), 176 - 189.

[6] W.J. LEVEQUE : *On Mahler's U numbers.* Journal of the London Mathematical Society **28** (1953), 220 - 229.

[7] W.J. LEVEQUE : Topics in number theory, Vol II. London, England (1956).

[8] K. MAHLER : *Zur Approximation der Exponentialfunktion und des Logarithmus I.* J. f.d. reine u. angew. Math. 166 (1932), 137 - 150.

[9] K. MAHLER : *Über eine Klassen - Einteilung der p-adischen Zahlen.* Mathematica (Leiden) 3 (1935), 177 - 185.

[10] O. PERRON : Irrationalzahlen, Zweite Auflage. Berlin (1939), 182 - 186.

[11] TH. SCHNEIDER : Einführung in die Transzendenten Zahlen. Berlin - Göttingen - Heidelberg (1957).

[12] E. WIRSING : *Approximation mit algebraischen Zahlen beschränkten Grades.* J. f.d. reine u. angew, Math. 206 (1961), 67 - 77.

ÖZET

Bu çalışmada kuvvetli bir Liouville sayısının cebirsel katsayılı tam ve rasyonel kombinezonları incelenerek bunların Mahler'in U_m alt sınıfına ait oldukları gösterilmektedir (Burada m, bu katsayıların belirttiği cebirsel sayı cisminin derecesini göstermektedir). Böylece $U_m (m = 1, 2, ...)$ Mahler alt sınıflarının hiçbirinin boş olmadığına dair ilk önce 1953 de LEVEQUE tarafından elde edilen sonucun yeni bir ispatı bulunmuş olmaktadır. Tam kombinezonlar halinde, Hensel'in p - adik sayılar cisminde yukarıkine benzer bir sonuç elde edilmektedir.

KARADENİZ TEKNİK ÜNİVERSİTESİ
TEMEL BİLİMLER FAKÜLTESİ
MATEMATİK BÖLÜMÜ
TRABZON, TÜRKİYE

(Manuscript received October 16, 1978, revised February 14, 1980)

Corrections to the foregoing paper

Please make the following corrections in the references quoted in the text :

page	line	wrong	right
41	2 and 20	GÖTİNG [1]	GÖTING [3]
44	7	LEVEQUE [1]	LEVEQUE [6]
51	18	P. ERDÖS [1]	P. ERDÖS [2]
66	8	KOKSMA [1]	KOKSMA [5]
72	8 and 19	K. MAHLER [2]	K. MAHLER [9]