UNIVALENT RATIONAL FUNCTIONS ON THE UNIT DISC

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In this paper we give a set of necessary and sufficient conditions for an analytic rational function defined in the unit disc to be univalent.

Let $D = \{z \in \mathbf{C} : |z| < 1\}$ be the unit disc in the complex plane $\mathbf{C}$, and let $w = h(z)$ be au analytic rational function in D. In this paper wc will give a set of necessary and sufficient conditions for h to be univalent in D.

It is clear that any univalent rational function h is of the form

$$h(z) = h'(0) \frac{z(1-\alpha_1 z) \dots (1-\alpha_{p-1} z)}{(1-\beta_1 z) \dots (1-\beta_p z)} + h(0) \tag{1}$$

where $|\alpha_i| \leqslant 1$, $|\beta_i| \leqslant 1$ and p is a positive integer.

This follows from the facts that h is analytic and takes $h(0)$ only once in D.

Therefore we assume that h is given by (1) and proceed to find our conditions. It follows from (1) that

$$\frac{h(z) - h(\zeta)}{z - \zeta} = h'(0) \frac{\sum_{i=1}^{p} A_i(\zeta) z^{p-i}}{\prod_{j=1}^{p} [(1-\beta_j z)(1-\beta_j \zeta)]}$$

where $A_i(\zeta)$ are polynomials in ζ, of degree $\leqslant p-1$. Note that $A_p(\zeta) = (1-\alpha_1 \zeta) \dots (1-\alpha_{p-1} \zeta)$. Hence $A_p(\zeta)$ is never zero when $|\zeta| < 1$. Letting $B_i(\zeta) = A_i(\zeta)/A_p(\zeta)$ the preceding eqnality can be written as

$$\frac{h(z)-h(\zeta)}{z-\zeta}=h'(0)\;A_p(\zeta)\;\frac{\sum\limits_{i=1}^{p}B_i(\zeta)\,z^{p-}}{\prod\limits_{j=1}^{p}\left[(1-\beta_j z)\,(1-\beta_j\zeta)\right]}\,.$$

From this equality we conclude that $h(z)$ is univalent in $|z| < 1$ if and only if $Q(z) = B_1(\zeta)\,z^{p-1} + \dots + B_{p-1}(\zeta)\,z + 1$ is never zero for all $|z| < 1$ and $|\zeta| < 1$. In other words, for $h(z)$ to he univalent in $|z| < 1$, it is necessary and sufficient that the roots of $Q(z)$ are outside of the unit disc for each $|\zeta| < 1$.

There are several methods dealing with this kind of problem. One of these is called the Hurwitz criterion. This criterion gives the necessary and sufficient conditions which the coefficients of an arbitrary polynomial (with real coefficients) must satisfy in order for its roots to lie in the left half plane. Our problem with $Q(z)$ can be reduced to this case. For this we define a new polynomial $P(z)$ in z by

$$P(z) = Q(z)\,.\,\overline{Q(\overline{z})} \tag{2}$$

with this definition it is clear that if all the roots of $Q(z)$ lie outside of the unit disc, the same will be true for the polynomial $P(z)$. Moreover, the coefficients of $P(z)$ are real. $z(\eta) = \dfrac{\eta - 1}{\eta + 1}$ maps the right half plane into the unit disc and the left half plane outside of the unit disc. Hence

$$R(\eta) = (\eta + 1)^{2(p-1)}\,P\!\left(\frac{\eta - 1}{\eta + 1}\right) \tag{3}$$

is a polynomial in η with real coefficients, all of roots lying on the left half plane if the roots of $Q(z)$ lie outside of the unit disc

Now let

$$\psi(\eta) = \frac{R(\eta) + R(-\eta)}{R(\eta) - R(-\eta)}\,. \tag{4}$$

Then we can write $\psi(\eta)$ as a finite fractional form. Namely,

(5) $$\psi(\eta) = C_1(\zeta)\,\eta + \cfrac{1}{C_2(\zeta)\eta + \cfrac{1}{C_3(\zeta)\,\eta + \dots + \cfrac{1}{C_{2(p-1)}(\zeta)\,\eta}}}.$$

Then necessary and sufficient conditions that $R(\eta)$ will have all of this roots on the left half plane are expressed by the statement that $C_i(\zeta) \geqslant 0$ for all $|\zeta| < 1$ and all $i = 1,\dots, 2(p-1)$, (Hurwitz criteria).

With the above notation we now have proved the following theorem :

Theorem 1. *Let $|\alpha_i| \leqslant 1$, $|\beta_i| \leqslant 1$ for $i = 1,\dots, p$ and let $h'(0)$ and $h(0)$ be arbitrary constants. Then $w = h(z)$, defined by (1), is univalent in $|z| < 1$ if and only if all the quantities $C_i(\zeta)$, defined by the equality (5), are positive functions of ζ on the unit disc $|\zeta| < 1$.*

In an almost trivial case, namely for $p = 2$ we will now test this method and obtain necessary and sufficient conditions on α_1, β_1, β_2 for $h(z)$ to be univalent. If $p = 2$ then

$$Q(z) = B_1(\zeta)\,z + 1$$

Using definitions (2) and (3) we obtain

$$B(\eta) = |1 + B_1(\zeta)|^2\,\eta^2 + 2(1 - |B_1(\zeta)|^2)\,\eta + |1 - B_1(\zeta)|^2$$

Hence we find

$$\psi(\eta) = \frac{|1 + B_1(\zeta)|^2\,\eta^2 + |1 - B_1(\zeta)|^2}{2(1 - |B_1(\zeta)|^2)\,\eta}$$

or

$$\psi(\eta) = \frac{|1 + B_1(\zeta)|^2}{2(1 - |B_1(\zeta)|^2)}\,\eta + \cfrac{1}{\cfrac{2(1 - |B_1(\zeta)|^2)}{|1 - B_1(\zeta)|^2}\,\eta}$$

From which it follows immediately that

$$C_1(\zeta) = \frac{|1 + B_1(\zeta)|^2}{2(1 - |B_1(\zeta)|^2)} \quad\text{and}\quad C_2(\zeta) = \frac{2(1 - |B_1(\zeta)|^2)}{|1 - B_1(\zeta)|^2}.$$

Now it is clear that $C_1(\zeta) \geqslant 0$ and $C_2(\zeta) \geqslant 0$ if and only if $|B_1(\zeta)| \leqslant 1$. After some calculation one finds that

$$B_1(\zeta) = \frac{\{\alpha_1 \beta_1 + \alpha_1 \beta_2 - \beta_1 \beta_2\} \zeta - \alpha_1}{1 - \alpha_1 \zeta}$$

It is easy to find out that $B_1(\zeta)$ maps the unit disc $|\zeta| < 1$, onto the disc of radius

$$r = \frac{|(\beta_1 - \alpha_1)(\alpha_1 - \beta_2)|}{1 - |\alpha_1|^2}$$

centered at

$$A = \frac{|\alpha_1|^2 (\beta_1 + \beta_2) - \bar{\alpha}_1 \beta_1 \beta_2 - \alpha_1}{1 - |\alpha_1|^2}.$$

Therefore $|B(\zeta)| \leqslant 1$ for all $|\zeta| < 1$, if and only if $|A| + r \leqslant 1$.

We have proven the following theorem :

Theorem 2. *Let α, β, γ lie in the unit disc $|z| < 1$, and let $h'(0)$, $h(0)$ be two arbitrary constants. Then*

$$h(z) = h'(0) \frac{z(1 - \alpha z)}{(1 - \beta z)(1 - \gamma z)} + h(0)$$

is univalent in $|z| < 1$ if and only if

$$\left|(\beta + \gamma)|\alpha|^2 - \bar{\alpha} \beta \gamma - \alpha\right| + \left|(\beta - \alpha)(\alpha - \gamma)\right| + |\alpha|^2 \leqslant 1.$$

Note that for $p = 3$, there are four conditions in Theorem 1. One can show that this number can be reduced by two. Without proof, we state this result as a seperate theorem.

Theorem 3. *Let* $\alpha_i (i = 1,2)$, $\beta_j (j = 1,2,3)$ lie in the unit disc $|z| < 1$, and let $h'(0)$ and $h(0)$ be two arbitrary constants. Then

$$h(z) = h'(0) \frac{z(1 - \alpha_1 z)(1 - \alpha_2 z)}{(1 - \beta_1 z)(1 - \beta_2 z)(1 - \beta_3 z)} + h(0)$$

is univalent in $|z| < 1$ if and only if the following two conditions hold for all $|\zeta| < 1$:

i) $Re\ \lambda(\zeta) \geqslant 0$

ii) $(Im\ \lambda(\zeta)\ \overline{\mu(\zeta)})^2 \leqslant 4\ Re\ \lambda(\zeta).\ Re\ \mu(\zeta)$, *where*

$$\lambda(\zeta) = \frac{1 + B_1(\zeta) + B_2(\zeta)}{1 - B_1(\zeta)} \quad \text{and} \quad \mu(\zeta) = \frac{1 + B_1(\zeta) - B_2(\zeta)}{1 - B_1(\zeta)}$$

For more information with respect to the Hurwitz criteria and a similar criteria the reader is referred to GUILLEMIN [2] and COHN [1].

If one is intereted only in necessary or sufficient conditions for $h(z)$, defined by (1), to he univalent in $|z| < 1$, there are several conditions. We only mention two here :

a) Let $\{w, z\}$ be the Schwarzian, then $w = h(z)$ is univalent on $|z| < 1$, if

$$\{w, z\} \leqslant \frac{2}{(1 - |z|^2)^2}$$

for all $|z| < 1$; (NEHARI [3], SCHIFFER and HAWLEY [4]).

b) If $w = h(z)$ is univalent on $|z| < 1$, then

$$\{w, z\} \leqslant \frac{6}{(1 - |z|^2)^2}$$

for all $|z| < 1$; (NEHARI [3]).

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ÖZET

Bu makalede birim dairede tanımlı rasyonel bir analitik fonksiyonun yalınkat olması için gerek ve yeter koşullar verilmektedir.

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