# ON AFFINE MOTION IN A RECURRENT FINSLER SPACE [+)]

A. KUMAR [*)]

Consider an *affine motion*, that is to say an infinitesimal point transformation for which the deformed space has the same affine connection as the original one, in a *recurrent* FINSLER space $F_n$, *i.e.* in a FINSLER space whose curvature tensor $H^i_{hjk}(x, \dot{x})$ satisfies the differential system

$$H^i_{hjk(l)} = \lambda_l \, H^i_{hjk},$$

where $\lambda_l$ is a non zero vector. We prove that if this motion is of the form

$$\text{(a)} \qquad \overline{x}^i = x^i + v^i(x)\, dt$$

with $v^i_{(j)} = p\delta^i_j$, $p = p(x)$, then $p(x)$ has to vanish and $v^i$ spans a field of parallel contravariant vectors. Futhermore, if a non-flat recurrent FINSLER space $F_n$ admits an affine motion $(a)$ such that $v^i = v^i(x)$ span a contra-field in $F_n$, the conditions

$$v^s \lambda^s = 0, \; H^i_{hjk}\, v^k = 0$$

are satisfied.

**1. Introduction.** Let us consider an $n$-dimensional FINSLER space $F_n$ [1] [1)] equipped with a fundamental metric function $F(x, \dot{x})$ positively homogeneous of degree one in its directional arguments. The fundamental metric tensor [2)] $g_{ij}(x, \dot{x}) \stackrel{\text{def}}{=} \frac{1}{2}\, \dot{\partial}_i \,.\, \dot{\partial}_j \,.\, F^2(x, \dot{x})$ of the space is symmetric in its

---

+) Communicated by Prof. Dr. RAM BEMARI on November 24, 1975.

*) In concluding this manuscript, the author appreciates very much the kind help from Dr. H. D. PANDE. The author's sincere thanks go to him.

1) The numbers in brackets refer to the references at the end of the paper.

2) $\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$ , $\partial_i \equiv \partial/\partial x^i$.

lower indices. The covariant derivative of a contravariant vector field $X^i(x, \dot{x})$ in the sense of BERWALD is given by

$$X^i_{(k)} = \partial_k X^i - \dot{\partial}_m X^i G^m_{rk} \dot{x}^r + X^h G^i_{hk}, \tag{1.1}$$

where $G^i_{hk}(x, \dot{x})$ are BERWALD's connection coefficients. The curvature tensor arising from the commutation formula involving the above covariant derivative is given by [3)]

$$H^i_{hjk}(x, \dot{x}) \overset{\text{def}}{=} 2 \{\partial_{[k} G^i_{j]h} - G^i_{rk[j} G^r_{k]} + G^i_{r[k} G^r_{j]h}\} \tag{1.2}$$

and satisfies

$$a) \quad H^i_{hjk} = - H^i_{hkj} \quad \text{and} \quad b) \quad (\dot{\partial}_s H^i_{hjk}) \dot{x}^s = 0. \tag{1.3}$$

Let us consider an infinitesimal transformation

$$\overline{x}^i = x^i + v^i(x)\, dt, \tag{1.4}$$

where $v^i(x)$ being a vector field defined over the domain of the space under consideration and $dt$ is an infinitesimal constant. In view of the above transformation and BERWALD's covariant derivative the LIE-derivative of any tensor field $T^i_j(x, \dot{x})$ is given by

$$\pounds_v T^i_j(x, \dot{x}) = T^i_{j(h)} v^h + (\dot{\partial}_h T^i_j) v^h_{(r)} \dot{x}^r + T^i_h v^h_{(j)} - T^h_j v^i_{(h)}. \tag{1.5}$$

We have the following commutation formulae:

$$\pounds_v (\dot{\partial}_l T^i_j) - \dot{\partial}_l (\pounds_v T^i_j) = 0 \tag{1.6}$$

and

$$(\pounds_v G^i_{jh})_{(k)} - (\pounds_v G^i_{kh})_{(j)} = \pounds_v H^i_{hjk} + 2 \dot{x}^l G^i_{rh[j} \pounds_v G^r_{k]l}. \tag{1.7}$$

An $n$-dimensional FINSLER space is called a recurrent FINSLER space if its curvature tensor $H^i_{hjk}$ satisfies the relation

$$H^i_{hjk(l)} = \lambda_l H^i_{hjk}, \tag{1.8}$$

where $\lambda_l$ is a non-zero vector.

3) $2 A_{[hk]} = A_{hk} - A_{kh}$, $2 A_{(hk)} = A_{hk} + A_{kh}$.

**2. Affine motion and its compatibility.** In view of the infinitesimal point teransformation (1.4) we have a deformed space with affine connetion $G^i_{jk} + (\pounds_v G^i_{jk})$ dt. If the original space and the deformed space have the same affine connection, the transformation is called affine motion of the space $F_n$. Such a motion may be, of course, considered in recurrent FINSLER space. In order that it be the case, it is necessary and sufficient that, we have

$$\pounds_v G^i_{jk} \equiv v^i_{(j)(k)} + H^i_{hjk} v^h + G^i_{sjk} v^s_{(r)} \dot{x}^r = 0. \tag{2.1}$$

In view of the commutation formula (1.7) and affine motion, we get necessarily

$$\pounds_v H^i_{hjk} = 0. \tag{2.2}$$

With respect to LIE-derivation, for any tensor, we have

$$(\pounds_v T^i_{jk(l)}) - (\pounds_v T^i_{jk})_{(l)} = T^m_{jk} \pounds_v G^i_{ml} - T^i_{mk} \pounds_v G^m_{jl} - T^i_{jm} \pounds_v G^m_{kl} - (\dot{\partial}_m T^i_{jk}) \pounds_v G^m_{lr} \dot{x}^r, \tag{2.3}$$

where we have used the commutation formula (1.6) and the fact that

$$\pounds_v \dot{x}^i = 0.$$

In view of the equations (2.1) and (2.2) using the commutation formula (2.3), for the curvature tensor $H^i_{hjk}(x, \dot{x})$, we get

$$\pounds_v (H^i_{hjk(l)}) = 0. \tag{2.4}$$

Continuing this process, we can see that the LIE-derivatives of the curvature tensor and of their successive covariant derivatives must be all zero. This fact suggests the following

Lemma: *In order that an affinely connected space admit a group $G_\lambda$ of affine motions, it is necessary and sufficient that there exist a positive number $K$ such that the first $K$ sets of equations (2.2), (2.4),... be compatible in the variables $v^i$ and $v^i_{(j)}$ and all such solutions satisfy the $(K+1)$-st set of the equations.*

The above lemma is due to M. S. KNEBELMAN [6] which gives us a certification of existence of affine motion. But in our case for an affine motion, we should have (2.1) and (2.2).

With the help of equations (1.8) and (2.2), we obtain the following set of relations one after another :

$$
(2.5)\qquad
\begin{aligned}
&\pounds_v H^i_{hjk} = 0\\
&\pounds_v (H^i_{hjk(l_1)}) = H^i_{hjk} \pounds_v \lambda_{l_1}\\
&\pounds_v (H^i_{hjk(l_1)(l_2)}) = H^i_{hjk} [\lambda_{l_2} \pounds_v \lambda_{l_1} + (\pounds_v \lambda_{l_1})_{(l_2)}]\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}
$$

where we have used the commutation formula (1.6).

3. **Contra-field in recurrent FINSLER space.** In view of the equation (1.5) the LIE-derivative of curvature tensor $H^i_{hjk}(x, \dot{x})$ is given by

$$
(3.1)\qquad \pounds_v H^i_{hjk} = H^i_{hjk(s)} v^s + (\dot{\partial}_s H^i_{hjk}) v^s_{(r)} \dot{x}^r - H^s_{hjk} v^i_{(s)} + H^i_{hjk} v^s_{(h)} + H^i_{hsk} v^s_{(j)} + H^i_{hjs} v^s_{(k)}
$$

which in view of the equations (1.8) and (2.2) reduces to

$$
(3.2)\qquad H^i_{hjk} \lambda_s v^s + (\dot{\partial}_s H^i_{hjk}) v^s_{(r)} \dot{x}^r - H^s_{hjk} v^i_{(s)} + H^i_{sjk} v^s_{(h)} + H^i_{hsk} v^s_{(j)} + H^i_{hjs} v^s_{(k)} = 0.
$$

If a recurrent $F_n$ admits an affine motion of the form

$$
(3.3)\qquad \overline{x}^i = x^i + v^i(x)\ dt, \quad v^i_{(j)} = p\delta^i_j,
$$

where $p(x) \neq 0$, then in view of (3.3) and (1.3) $b$, the equation (3.2) reduces to

$$
(3.4)\qquad (v^s \lambda_s + 2p)\ H^i_{hjk} = 0.
$$

Since $H^i_{hjk} \neq 0$, from this, we get

$$p(x, \dot{x}) = -\tfrac{1}{2} v^s \lambda_s . \tag{3.5}$$

Hence the motion (3.3) takes the following form

$$\overline{x}^i = x^i + v^i(x)\ dt, \quad v^i_{(j)} = -\tfrac{1}{2} v^s \lambda_s \delta^i_j . \tag{3.6}$$

Substituting the latter of (3.6) in (2.1) and using the homogeneity property of $G^i_{hjk}(x, \dot{x})$, we get

$$H^i_{jkh} v^h = -\tfrac{1}{4} (2\lambda_{m(k)} - \lambda_m \lambda_k) v^m \delta^i_j . \tag{3.7}$$

In view of the equation (1.8), the BIANCHI identity

$$H^i_{hjh(l)} + H^i_{hkl(j)} + H^i_{hlj(k)} = 0 \tag{3.8}$$

yields

$$\lambda_l H^i_{hjk} + \lambda_j H^i_{hkl} + \lambda_k H^i_{hlj} = 0. \tag{3.9}$$

Multiplying this identity by $v^l$ and summing up with respect to $l$, we shall get

$$H^i_{hjk} \lambda_l v^l = -H^i_{hkl} v^l \lambda_j + H^i_{hjl} v^l \lambda_k , \tag{3.10}$$

where we have used the equation (1.3) $a$. Introducing (3.7) in the right hand side of (3.10), we obtain

$$H^i_{hjk} \lambda_l v^l = \tfrac{1}{2} \{\lambda_{l(j)} \lambda_k - \lambda_{l(k)} \lambda_j\} v^l \delta^i_h . \tag{3.11}$$

With the help of the identity

$$H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0, \tag{3.12}$$

we obtain

$$H^i_{hjk} \lambda_l v^l + H^i_{jkh} \lambda_l v^l + H^i_{khj} \lambda_l v^l = 0. \tag{3.13}$$

Substituting (3.11) in (3.12), we get

$$v^l \,[\delta^i_h(\lambda_{l(j)}\,\lambda_k - \lambda_{l(k)}\,\lambda_j) + \delta^i_j(\lambda_{l(k)}\,\lambda_h - \lambda_{l(h)}\,\lambda_k) + \delta^i_k(\lambda_{l(h)}\,\lambda_j - \lambda_{l(j)}\,\lambda_h)] = 0. \tag{3.14}$$

Contracting with respect to the indices $i$ and $h$, from this, it follows that

$$v^l \,[(n-2)\,(\lambda_{l(k)}\,\lambda_j - \lambda_{l(j)}\,\lambda_k)] = 0, \tag{3.15}$$

so, if $n \geq 3$, we get

$$v^l \,\{\lambda_{l(k)}\,\lambda_j - \lambda_{l(j)}\,\lambda_k\} = 0. \tag{3.16}$$

Comparing the above result with (3.11), we can see that

$$H^i_{hjk}\,v^l\,\lambda_l = 0. \tag{3.17}$$

Since $H^i_{kjk} \neq 0$, we have

$$\lambda_l\,v^l = 0. \tag{3.18}$$

Consequently with the help of equations (3.5) and (3.18), we get

$$p(x) = 0. \tag{3.19}$$

Thus, we have

**Theorem (3.1):** *If a non flat recurrent* FINSLER *space* $(n \geq 3)$ *admits an affine motion of the form* (3.3), $p(x)$ *has to vanish; that is to say that* $v^i$ *has to span a field of parallel contravariant vectors.*

We shall now try to find a necessary and sufficient condition in the case of a special affine motion of the form

$$\bar{x}^i = x^i + v^i(x)\,dt, \quad v^i_{(j)} = 0. \tag{3.20}$$

In view of the above condition, the equation (2.1) of motion becomes

$$\pounds_v\,G^i_{jk} = -H^i_{jkh}\,v^h = 0 \tag{3.21}$$

and the integrability condition (2.1) becomes

(3.22) $$\pounds_v H^i_{hjk} = v^s \lambda_s H^i_{hjk} = 0.$$

So, owing to the non-flatness of the space, from (3.22), we get

(3.23) $$v^s \lambda_s = 0.$$

In view of equations (3.21) and (3.23), we conclude that

*In order that a recurrent* FINSLER *space admit an affine motion of the form* (3.20), *it is necessary that, we have*

(3.24) $$v^s \lambda_s = 0, \; H^i_{jkh} v^h = 0.$$

Conversely, if we have (3.24), using the identity

(3.25) $$H^i_{jkh} + H^i_{khj} + H^i_{hjk} = 0$$

and the latter part of (3.24), we obtain

(3.26) $$H^i_{hjk} v^h = 0.$$

From the general theory of parallel vectors [7], $v^i$ determines a contrafield in a recurrent FINSLER space, say $v^i_{(j)} = 0$, when and only when it satisfies the equation (3.25).

Thus we may regard $v^i$ as a contrafield in recurrent FINSLER space. This gives rise to

Theorem (3.2). *If a non flat recurrent* FINSLER *space admits an infinitesimal affine motion* $\overline{x}^i = x^i + v^i(x)\, dt$ *such that the vector* $v^i(x)$ *spans a contrafield in recurrent* $F_n$, *then it is necessary and sufficient that*

$$v^s \lambda_s = 0 \text{ and } H^i_{hjk} v^h = 0$$

*be valid.*

## REFERENCES


[2] RUND, H. : The differential geometry of FINSLER space, SPRINGER VERLAG, BERLIN, (1959).

[2] RUSE, H. S. : *A classification of $K^*$-space*, Proc. London Math. Soc., (2), **53** (1951), 212 - 229.

[2] WALKER, A. G. : *On* RUSE's *spaces of recurrent curvature*, ibid., (2), 52 (1950), 36 - 64.

[4] WONG, Y. C. : *A class of non-Riemannian $K^*$-spaces*, ibid., (3), **3** (1953), 118-128.

[5] KNEBLMAN, M. S. : *On the equations of motion in a Riemannian space*, Bull. Amer. Math. Soc., **51** (1945), 682 - 685.

[6] TAKANO, K. : *Affine motion in non-Riemannian $K^*$-spaces*, Tensor, N. S. (2), **11** (1961), 137 - 143.

[7] EISENHART, L. P. : Non-Riemannian Geometry, NEW YORK (1972), 18 - 22.


DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GORAKHPUR
GORAKHPUR - 273001 (U. P.)
INDIA

*(Manuscript received on November 24, 1975)*

## ÖZET


*İndirgenmeli* bir FINSLER uzayında, yani, $\lambda_l$ sıfır olmayan bir vektör olmak üzere, $H^i_{hjk(l)}(x, \dot{x})$ eğrilik tensörü

$$H^i_{hjk(l)} = \lambda_l \, H^i_{hjk}$$

diferansiyel sistemini sağlayan bir FINSLER uzayında bir *afin hareket*, dönüştürülmüş uzayın afin bağımlılığını ilk uzayın afin bağımlılığına eşit bırakan bir infinitezimal nokta dönüşümüdür. $v^i_{(j)} = p\delta^i_j$, $p = p(x)$ olmak üzere, afin hareket

$$(a) \qquad \bar{x}^i = x^i + v^i(x)\, dt$$

şeklinde ise, $p(x)$ in sıfır olmak zorunda olduğu, yani $v^i$ nin paralel kontravaryant vektörlerden oluşan bir alam doğurduğu kanıtlanmıştır. Üstelik, düz olmayan bir indirgenmeli $F_n$ FINSLER uzayında, $v^i(x)$ ler $F_n$ de bir kontra-alan doğuracak şekilde $(a)$ gibi bir afin hareket verilecek olursa

$$v^s \lambda_s = 0, \quad H^i_{hjk}\, v^k = 0$$

koşulları gerçeklenir.