ON SOLUTIONS OF COUPLED ELECTROMAGNETIC AND ZERO-REST-MASS SCALAR FIELDS

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The purpose of this paper is to study the solutions of the field equations corresponding to a space time represented by the metric

$$ds^2 = -A\,dx^2 - 2D\,dx\,dy - B\,dy^2 - (C - E)\,dz^2 - 2E\,dz\,dt + (C + E)\,dt^2,$$

where A, B and D are functions of the single variable $Z = Z(z - t)$, C is a function of z and t and E is a function of x, y, z and t.

1. Introduction. TAKENO [1], LAL and PRASAD [2] have considered the relativistic field equation for regions of space containing electromagnetic fields but no matter in plane and cylindrical symmetries and succeeded in finding the wave solutions. LAL, SINGH [3] and JANIS, NEWMAN and WINICOUR [4] have considered the solutions of the field equation cf general relativity containing zero-rest-mass scalar fields. RAO, ROY and TEWARI [5] and STEPHENSON [6] have considered the general relativistic field equations for regions of space containing, in addition to an electromagnetic field, a scalar meson field V associated with a meson of rest-mass μ and have found a class of exact solutions for the EINSTEIN-ROSEN metric while STEPHENSON has considered static solutions in spherical symmetry. The field equations coupled with electromagnetic and scalar fields with meson of rest mass $\mu = 0$, are given by

$$G_{ij} \equiv R_{ij} - \tfrac{1}{2} g_{ij}\, R = -8\pi (T_{ij} + M_{ij}), \tag{1.1}$$

$$g^{ij}\, V_{;ij} = 0, \tag{1.2}$$

where R_{ij} and R are the RICCI tensor and scalar curvature of the space-time under consideration and V is the zero-rest-mass scalar field. T_{ij} and M_{ij} are the electromagnetic energy-momentum tensor and energy-momentum tensor of zerorest-mass scalar fields, respectively defined by

$$T_{ij} = - F_{is} F_j^s + \tfrac{1}{4} g_{ij} F_{sp} F^{sp}, \tag{1.3}$$

$$M_{ij} = V_{,i} V_{,j} - \tfrac{1}{2} g_{ij}(V_{,l} V_{,m} g^{lm}), \tag{1.4}$$

where F_{ij} is the anti-symmetric electromagnetic field tensor and satisfies the generalized MAXWELL equations

$$\text{(1.5)} \quad a) \quad F_{ij;k} + F_{jk;i} + F_{ki;j} = F_{ij,k} + F_{jk,i} + F_{ki,j} = 0,$$

$$b) \quad F^{ij}_{;j} = 0.$$

A semi-colon (;) denotes covariant differentiation with respect to the CHRISTOFFEL symbols $\{^{k}_{ij}\}$ and a comma (,) followed by an index i denotes partial differentiation with respect to x^i.

In this paper we propose to consider the field equations (1.1), (1.2) and (1.5) and proceed to find the solutions in the space-time represented by the metric

$$ds^2 = -A\,dx^2 - 2D\,dx\,dy - B\,dy^2 - (C-E)\,dz^2 - 2E\,dz\,dt + (C+E)\,dt^2, \tag{1.6}$$

where A, B, D are functions of a single variable $Z = Z(z - t)$, C and E are arbitrary functions of (z, t) and (x, y, z, t) respectively.

2. Calculation of G_{ij}, T_{ij} and M_{ij}. The non-vanishing components of the contravariant tensor g^{ij} for the line-element (1.6) are given by

$$\text{(2.1)} \quad \begin{aligned} &g^{11} = - B/m,\ g^{22} = - A/m,\ g^{33} = - (C + E)/C^2, \\ &g^{44} = (C - E)/C^2,\ g^{12} = D/m,\ g^{34} = - E/C^2, \end{aligned}$$

where $m = AB - D^2$. The CHRISTOFFEL symbols of the second kind $\{^{k}_{ij}\}$ have been calculated and their non-vanishing components are given by

$$\left\{\begin{matrix}3\\11\end{matrix}\right\} = \left\{\begin{matrix}4\\11\end{matrix}\right\} = -\bar{A}/2C, \left\{\begin{matrix}3\\12\end{matrix}\right\} = \left\{\begin{matrix}4\\12\end{matrix}\right\} = -\bar{D}/2C,$$

$$\left\{\begin{matrix}3\\14\end{matrix}\right\} = \left\{\begin{matrix}4\\14\end{matrix}\right\} = -\left\{\begin{matrix}3\\13\end{matrix}\right\} = -\left\{\begin{matrix}4\\13\end{matrix}\right\} = E_1/2C,$$

$$\left\{\begin{matrix}1\\13\end{matrix}\right\} = -\left\{\begin{matrix}1\\14\end{matrix}\right\} = (\bar{A}B - \bar{D}D)/2m, \left\{\begin{matrix}2\\13\end{matrix}\right\} = -\left\{\begin{matrix}2\\14\end{matrix}\right\} = (A\bar{D} - \bar{A}D)/2m,$$

$$\left\{\begin{matrix}3\\22\end{matrix}\right\} = \left\{\begin{matrix}4\\22\end{matrix}\right\} = -\bar{B}/2C,$$

$$\left\{\begin{matrix}1\\23\end{matrix}\right\} = \left\{\begin{matrix}1\\24\end{matrix}\right\} = (B\bar{D} - \bar{B}D)/2m, \left\{\begin{matrix}2\\23\end{matrix}\right\} = -\left\{\begin{matrix}2\\24\end{matrix}\right\} = (A\bar{B} - D\bar{D})/2m,$$

$$\left\{\begin{matrix}3\\24\end{matrix}\right\} = \left\{\begin{matrix}4\\24\end{matrix}\right\} = -\left\{\begin{matrix}3\\23\end{matrix}\right\} = -\left\{\begin{matrix}4\\23\end{matrix}\right\} = E_2/2C, \tag{2.2}$$

$$\left\{\begin{matrix}1\\33\end{matrix}\right\} = \left\{\begin{matrix}1\\44\end{matrix}\right\} = -\left\{\begin{matrix}1\\34\end{matrix}\right\} = (BE_1 - DE_2)/2m,$$

$$\left\{\begin{matrix}2\\33\end{matrix}\right\} = \left\{\begin{matrix}2\\44\end{matrix}\right\} = -\left\{\begin{matrix}2\\34\end{matrix}\right\} = (AE_2 - DE_1)/2m,$$

$$\left\{\begin{matrix}3\\33\end{matrix}\right\} = S + C_3/2C - E_3/2C, \left\{\begin{matrix}4\\33\end{matrix}\right\} = S + C_4/2C - E_3/C - E_4/2C,$$

$$\left\{\begin{matrix}3\\44\end{matrix}\right\} = S + C_3/2C + E_3/2C + E_4/C, \left\{\begin{matrix}4\\44\end{matrix}\right\} = S + C_4/2C + E_4/2C,$$

$$\left\{\begin{matrix}3\\34\end{matrix}\right\} = -S + C_4/2C - E_4/2C, \left\{\begin{matrix}4\\34\end{matrix}\right\} = -S + C_3/2C + E_3/2C,$$

where $S = E(E_3 + E_4 + C_3 - C_4)/2C^2$ and suffixes 1,2,3 and 4 attached to C and E and a bar over a function denote their partial derivatives with respect to x, y, z, t and Z respectively. The non-vanishing components of the RICCI tensor R_{ij} and the scalar curvature tensor R are given by

$$R_{13} = - R_{14} = (E_{13} + E_{14})/2C,$$

$$R_{23} = - R_{24} = (E_{23} + E_{24}(/2C,$$

(2.3) $$R_{33} = M - N(C - E)/C^2,$$

$$- R_{34} = - R_{43} = M + NE/C^2,$$

$$R_{44} = M + N(C + E)/C^2,$$

where

$$M = \{\overline{\overline{m}} = \overline{m}^2/2m - \overline{m}(C_3 - C_4)/2C - (\overline{A}\overline{B} - \overline{D}^2) - \overline{m}(E_3 + E_4)/2C$$

$$- (BE_{11} - 2DE_{12} + AE_{22})\}/2m,$$

$$2N = (C_3^2 - C_5^2)/C + (C_3 + C_4)(E_3 + E_4)/C - (C_{33} - C_{44}) - E_{33} - E_{44} - 2E_{34},$$

and

(2.4) $$R = g^{ij} R_{ij} = 2N/C^2.$$

Using the values of g^{ij} from (2.1), R_{ij} from (2.3) and R from (2.4), the non-vanishing components of G_{ij} are given by

$$G_{11} = (A/C^2)\,N,\ G_{12} = (D/C^2)\,N,\ G_{22} = (B/C^2)\,N,$$

$$G_{13} = - G_{14} = (E_{13} + E_{14})/2C,$$

(2.5)

$$G_{23} = - G_{24} = (E_{23} + E_{24})/2C,$$

$$G_{33} = - G_{34} = - G_{43} = G_{44} = M.$$

Taking the components of electromagnetic potentials as functions of x, y, Z and considering the transverse electromagnetic wave propagating along positive direction of the z - axis, the components of F_{ij} are obtained and are given by

$$F_{12} = F_{34} = 0,\ F_{13} = -F_{14} = -\sigma, \quad F_{23} = -F_{24} = \rho, \tag{2.6}$$

under the condition

$$\rho_{,1} + \sigma_{,2} = 0. \tag{2.7}$$

The non-vanishing components of T_{ij} given hy (1.3) are as follows:

$$T_{33} = -T_{34} = -T_{43} = T_{44} = (A\rho^2 + 2D\rho\sigma + B\sigma^2)/m. \tag{2.8}$$

Now considering the scalar meson field V as any function of $Z = Z(z - t)$, the components of M_{ij} are calculated and the non-vanishing components are

$$M_{33} = -M_{34} = -M_{43} = M_{44} = \bar{V}^2. \tag{2.9}$$

3. Solutions of the field equations (1.1), (1.2) and (1.5). Substituting the values of G_{ij} from (2.5), T_{ij} from (2.8) and M_{ij} from (2.9) in the equation (1.1) we have

$$(A/C^2)\,N = (D/C^2)\,N = (B/C^2)\,N = 0, \tag{3.1}$$

$$E_{13} + E_{14} = E_{33} + E_{24} = 0, \tag{3.2}$$

$$M = -8\pi\,\{\bar{V}^2 + (A\rho^3 + 2D\rho\sigma + B\sigma^2)/m\}. \tag{3.3}$$

The equations (3.1) give

$$N = 0, \tag{3.4}$$

and equations (3.2) on integration yield the four forms of E as below:

$$\begin{aligned} F &= (x + y + z)\,\phi_1(z - t) + \phi(x, y)\,\phi_2(z - t), \\ F &= (x + y + t)\,\phi_1(z - t) + \phi(x, y)\,\phi_2(z - t), \\ E &= \phi(z, y, z - t) + z\,\phi_1(z - t), \\ E &= \phi(x, y, z - t) + t\,\phi_1(z - t). \end{aligned} \tag{3.5}$$

Using the form E given by (3.5) in (3.4), the equation (3.4) reduces to

$$(3.6) \qquad (C_3^2 - C_4^2)/C + (C_3 + C_4)(E_3 + E_4)/C - (C_{33} - C_{44}) = 0.$$

Solving the equations (1.2) and (1.5) *a*) it is found that they are satisfied identically by the components of F_{ij} given by (2.6) with (2.7) and $V = V(z-t)$ while the equation (1.5) *b*) is satisfied subject to the condition

$$(3,7) \qquad (A\rho_{,2} - B\sigma_{,1}) - D(\rho_{,1} - \sigma_{,2}) = 0.$$

Thus, the solutions of the field equations of coupled electromagnetic and zero-rest-mass scalar fields are composed of the g_{ij} as given by (1.6), the F_{ij} as given by (2.6) with (2.7) and $V = V(z-t)$ under the conditions (3.3), (3.5), (3.6) and (3.7). We can say that the equations (3.5) and (3.6) restrict the forms of C and E and the equations (3.3) is only one condition for eight functions and we will get infinitely many sets of $\{(A, B, C, D, V), (\rho, \sigma, E)\}$ satisfying the relation (3.3) and it admits solutions with great arbitrariness under the restrictions of the forms of C and E given by (3.5) and (3.6). It is worth mentioning that the solutions found by TAKENO [7] are obtainable from the solutions obtained in this paper if we consider the functional forms of $A, B, C, D, E,$ ρ and σ as taken by him and put $\bar{V} = 0$ in equation (3.3).

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ÖZET

A, B ve D fonksiyonları tek $Z = Z(z - t)$ değişkenin fonksiyonları, C, z ve t değişkenlerinin bir fonksiyonu ve E de x, y, z ve t değişkenlerinin bir fonksiyonunu göstermek üzere,

$$ds^2 = -A\,dx^2 - 2D\,dx\,dy - B\,dy^2 - (C - E)\,dz^2 - 2E\,dz\,dt + (C + E)\,dt^2,$$

metriği ile veriien bir uzay zaman evrenindeki alan denklemlerinin çözümleri incelenmektedir.