# ON SOME EXTENSION THEOREMS CONCERNING GENERALIZED CAUCHY FUNCTIONAL EQUATIONS

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ACZÉL has shown that if $S$ is a sub-semigroup of the group $G$, such that

$$x \in G \Rightarrow x \in S \text{ or } x' \in S \text{ (where } x' \text{ is the inverse of } x)$$

then every homomorphism $f$ of $S$ into a group $H$ can be extended in a unique way to a homomorphism $g$ of $G$ into $H$ such that $f$ and $g$ coincide over $S$ and

$$\text{(f)} \qquad g(xy) = g(x)g(y)$$

for all $x, y \in G$.

The object of this paper is to evolve suitable techniques to extend an homomorphism $f$ from a subgroup $S$ of the group $G$ into a group $H$ to a group homomorphism $g$ of $G$ into $H$, such that $f$ and $g$ coincide on $S$ and the functional equation (f), or some analogue of this equation, hold.

**1. Introduction.** While studying some extensions of certain homomorphisms of subsemigroups to homomorphisms of groups, J. ACZÉL et al. [3] proved the following theorem :

Theorem 1. *Let $S$ be a subsemigroup of a group $G$ such that for each element $x$ of $G$, different from the identity element of $G$, either $x \in S$, or $x' \in S$ (or both) where $x'$ denotes the inverse of $x$. Then, every homomorphism $f$ of $S$ into a group $H$ can be extended in a unique way to a homomorphism $g$ of $G$ into $H$ such that*

$$g(x) = f(x), \qquad \forall\, x \in S$$

*and*

$$g(xy) = g(x)\, g(y), \qquad x \in G,\ y \in G.$$

It is clear that if $S$ is a subgroup of the group $G$, then $x \in S \Leftrightarrow x' \in S$. Hence the condition that for each element $x$ of $G$, different from the identity element of $G$, either $x \in S$ or $x' \in S$ (or both) no longer holds. Accordingly, the methods developed in [3] do not serve our purpose, if we want to extend the homomorphism $f$ of the subgroup $S$ into $H$ to a group homomorphism $g$ of $G$ into $H$ such that $g(x) = f(x)$ for all $x \in S$ and $g(xy) = g(x)\, g(y)$ for all $x, y \in G$.

The object of this paper is to study the problem of extending homomorphisms of subgroups to homomorphisms of groups. Not every subgroup homomorphism can be extended to a group homomorphism. However, in certain cases, it is possible to extend a group homomorphism to a group homomorphism. To demonstrate this, the authors have restricted to THIELMAN's functional equations.

2. The Sets $\Delta_n$, $\Gamma_n$ and $\Delta_n^*$. Let $R = (-\infty, \infty)$ and $E \subset R$. Following CHEVALLEY [5], a subset $E$ of $R$ is said to be stable with respect to the binary law of composition $\tau^n$ if $x \in E, y \in E \Rightarrow x\tau y \in E$. Let us define the disjoint subsets $\Delta_n$ and $\Gamma_n$ of $R$ as follows :

$$\Delta_n = \left\{ x \in R : \quad x > -\frac{1}{n} \right\}, \qquad n > 0,$$

$$\Gamma_n = \left\{ x \in R : \quad x < -\frac{1}{n} \right\}, \qquad n > 0.$$

If $\tau^n$ denotes the ordinary arithmetic addition, then $\Delta_n$ is not stable. For example, if $n = 1$, $x = -\frac{7}{8}$, $y = -\frac{3}{8}$, then $x + y = -\frac{10}{8} < -1$ and thus $x + y \notin \Delta_1$. However, if we consider the family of binary operations $\tau^n$, $n > 0$, defined by

$$\text{(A)} \qquad x\tau^n y = x + y + nxy,$$

where, on the right hand side of (A), we have ordinary arithmetic addition and multiplication, then $(\Delta_n, \tau^n)$ is a commutative group with real number

0 as the identity element. But $\Gamma_n$ is not stable with respect to the binary operation $\tau^n$ because $x < -\frac{1}{n}$, $y < -\frac{1}{n}$ implies that $x\tau^n y > -\frac{1}{n}$, $n > 0$.

Let $\Delta_n^* = \Delta_n \cup \Gamma_n$, $n > 0$. Obviously, $\Delta_n^* = R - \left\{-\frac{1}{n}\right\}$ and $(\Delta_n^*, \tau^n)$ is a commutative group of which $(\Delta_n, \tau^n)$ is a proper subgroup. Clearly, $\Gamma_n$ also denotes the set of those points which belong to the group $(\Delta_n^*, \tau^n)$ but not to the subgroup $(\Delta_n, \tau^n)$. The following can be easily derived by making use of (A).

(a) Denoting by $x'$, the inverse of $x \in (\Delta_n, \tau^n)$, it can be easily seen that $x' = -\frac{x}{1+nx}$ and further $x \in \Gamma_n \Leftrightarrow x' \in \Gamma_n$.

(b) $x \in (\Delta_n, \tau^n)$, $y \in (\Delta_n, \tau^n) \Rightarrow x\tau^n y \in (\Delta_n, \tau^n)$ *and* $x\tau^n y' \in (\Delta_n, \tau^n)$.

(c) $x \in \Gamma_n$, $y \in (\Delta_n, \tau^n) \Rightarrow x\tau^n y \in \Gamma_n$ *and* $x\tau^n y' \in \Gamma_n$.

(d) $x \in (\Delta_n, \tau^n)$, $y \in \Gamma_n \Rightarrow x\tau^n y \in \Gamma_n$ *and* $x\tau^n y' \in \Gamma_n$.

(e) $x \in \Gamma_n$, $y \in \Gamma_n \Rightarrow x\tau^n y \in (\Delta_n, \tau^n)$ *and* $x\tau^n y' \in (\Delta_n, \tau^n)$.

(f) $x \in \Delta_n \Leftrightarrow \left(-x - \frac{2}{n}\right) \in \Gamma_n$.

(g) $$\begin{cases} -(x\tau^n y) - \frac{2}{n} = x\tau^n\left(-y - \frac{2}{n}\right), & \text{if } x \in (\Delta_n, \tau^n),\ y \in \Gamma_n, \\ \qquad = \left(-x - \frac{2}{n}\right)\tau^n y, & \text{if } x \in \Gamma_n,\ y \in (\Delta_n, \tau^n). \end{cases}$$

(h) $x\tau^n y = \left(-x - \frac{2}{n}\right)\tau^n\left(-y - \frac{2}{n}\right)$, $\quad x \in (\Delta_n^*, \tau^n)$, $y \in (\Delta_n^*, \tau^n)$.

From the above observations, it is clear that for all $x \in (\Delta_n^*, \tau^n)$, $y \in (\Delta_n^*, \tau^n)$, the elements $x\tau^n y$ and $x\tau^n y'$ belong simultaneously either to $(\Delta_n, \tau^n)$ or to the set $\Gamma_n$.

**3. Generalized CAUCHY Functional Equation.** THIELMAN [4] discussed the functional equations

$$f_n(x+y+\mathrm{n}xy) = g_n(x) + h_n(y), \qquad x \in \Delta_n,\ y \in \Delta_n \tag{1}$$

and

$$f_n(x+y+nxy) = g_n(x)\, h_n(y), \qquad x \in \Delta_n,\ y \in \Delta_n, \tag{2}$$

where $f_n$, $g_n$ and $h_n$ are real-valued continuous functions with domain $\Delta_n$. We shall consider the following more general functional equation

$$f_n(x\tau^n y) = g_n(x)\, h_n(y), \tag{3}$$

in which the functions $f_n$, $g_n$ and $h_n$ are defined on the subgroup $(\Delta_n, \tau^n)$ and they take their values in an arbitrary group $\mathscr{E}$ which contains no zero element, that is, there does not exist any element $b \in \mathscr{E}$ such that

$$bx = xb = b, \qquad \text{for all } x \in \mathscr{E}.$$

It should be noted that on the right hand side of (3), $g_n(x)\, h_n(y)$ is to be computed in accordance with the group operation in $\mathscr{E}$.

Since every group is a semigroup, following the method of J. ACZÉL [2], the theorem given below can be easily proved :

**Theorem 2.** *The most general solutions of* (3) *among the functions* $f_n, g_n, h_n$ *mapping the commutative subgroup* $(\Delta_n, \tau^n)$ *into an arbitrary group* $\mathscr{E}$, *containing no zero element, are given by*

$$f_n(x) = g_n(0)\, k_n(x)\, h_n(0),\ \ g_n(x) = g_n(0)\, k_n(x),\ \ h_n(x) = k_n(x)\, h_n(0),\ \ x \in (\Delta_n, \tau^n), \tag{4}$$

*where* $k_n$ *is a homomorphism of* $(\Delta_n, \tau^n)$ *into* $\mathscr{E}$ i,e ;

$$k_n(x\tau^n y) = h_n(x)\, h_n(y), \qquad x, y \in (\Delta_n, \tau^n). \tag{5}$$

It may be noted that in the above theorem, it is not assumed that $\mathscr{E}$ is an abelian group. Also, in (3), $f_n$, $g_n$ and $h_n$ are just functions (not necessarily homomorphisms) and in (4), $g_n(0)$ and $h_n(0)$ may be assigned arbitrary values in $\mathscr{E}$.

**4. Extension Theorems Concerning (3).** In § 3, we have shown that the functional equation (3) admits of solutions of the form (4). Now, our object is to extend these solutions. Our method will be to obtain an extension $K_n: (\Delta_n^*, \tau^n) \to \mathscr{E}$ of the subgroup homomorphism $k_n: (\Delta_n, \tau^n) \to \mathscr{E}$ satisfying (5) and such that

$$K_n(x\tau^n y) = K_n(x)\, K_n(y), \qquad \text{for all } x, y \in (\Delta_n^*, \tau^n) \tag{6}$$

and

$$K_n(x) = k_n(x), \qquad \text{for all } x \in (\Delta_n, \tau^n) \tag{7}$$

and then define the extensions $F_n$, $G_n$ and $H_n$ of $f_n$, $g_n$ and $h_n$ respectively such that

$$F_n(x\tau^n y) = G_n(x)\, H_n(y), \qquad \text{for all } x, y \in (\Delta_n^*, \tau^n) \tag{8}$$

and

$$F_n(x) = f_n(x),\ G_n(x) = g_n(x),\ H_n(x) = h_n(x), \qquad \forall\, x \in (\Delta_n, \tau^n). \tag{9}$$

Theorem **3.** *Let $k_n$ be a subgroup homomorphism of $(\Delta_n, \tau^n)$ into the group $\mathscr{E}$. Then, to each $\lambda \in \mathscr{E}$ with $\lambda\lambda = e$, the identity element of $\mathscr{E}$, there exists a group homomorphism $K_{n,\lambda}$ of $(\Delta_n^*, \tau^n)$ into $\mathscr{E}$ such that $K_{n,\lambda}$ is an extension of $k_n$.*

Proof. Define $K_{n,\lambda}$ as follows :

$$K_{n,\lambda}(x) = \begin{cases} k_n(x) & \text{if } x \in (\Delta_n, \tau^n), \\ \lambda k_n\left(-x - \dfrac{2}{n}\right), & \text{if } x \in \Gamma_n, \end{cases} \tag{10}$$

where $\lambda \in \mathscr{E}$ commutes with each element of the range of $k_n$ and further

(11) $$\lambda\lambda = e, \text{ the identity element of } \mathscr{E}.$$

Since $\lambda$ commutes with each element of the range of $k_n$, therefore,

(12) $$\lambda k_n(x) = k_n(x)\,\lambda, \qquad \text{for all } x \in (\Delta_n, \tau^n).$$

Also, we know that $x \in \Gamma_n \Rightarrow \left(-x-\frac{2}{n}\right) \in \Delta_n$ so that $k_n\left(-x-\frac{2}{n}\right)$ belong to the range of $k_n$ and consequently

(13) $$\lambda k_n\left(-x-\frac{2}{n}\right) = k_n\left(-x-\frac{2}{n}\right)\lambda, \qquad \text{for all } x \in \Gamma_n.$$

We discuss the following four cases :

*Case (i).* $x \in (\Delta_n, \tau^n)$, $y \in (\Delta_n, \tau^n)$.

In this case, $x\tau^n y \in (\Delta_n, \tau^n)$ and thus $x\tau^n y \in (\Delta_n^*, \tau^n)$. Hence $K_{n,\lambda}$ satisfies (6) obviously.

*Case (ii).* $x \in (\Delta_n, \tau^n)$, $y \in \Gamma_n$.

In this case, because of (d), $x\tau^n y \in \Gamma_n$. Hence, we have

$$K_{n,\lambda}(x\tau^n y) \overset{(10)}{=} \lambda k_n\left[-(x\tau^n y)-\frac{2}{n}\right] \overset{(g)}{=} \lambda k_n\left[x\tau^n\left(-y-\frac{2}{n}\right)\right]$$

$$\overset{(5)}{=} \lambda k_n(x)\, k_n\left(-y-\frac{2}{n}\right) \overset{(12)}{=} k_n(x)\,\lambda k_n\left(-y-\frac{2}{n}\right) \overset{(10)}{=} K_{n,\lambda}(x)\, K_{n,\lambda}(y).$$

*Case (iii).* $x \in \Gamma_n$, $y \in (\Delta_n, \tau^n)$.

The proof is similar to that of the case (*ii*).

*Case (iv).* $x \in \Gamma_n$, $y \in \Gamma_n$.

In this case, by (*e*), $x\tau^n y \in (\Delta_n, \tau^n)$. Hence, we have

$$K_{n,\lambda}(x\tau^n y) \overset{(10)}{=} k_n(x\tau^n y) \overset{(h)}{=} k_n\left[\left(-x-\frac{2}{n}\right)\tau^n\left(-y-\frac{2}{n}\right)\right]$$

$$\overset{(5)}{=} k_n\left(-x-\frac{2}{n}\right) k_n\left(-y-\frac{2}{n}\right)$$

$$\overset{(11)}{=} k_n\left(-x-\frac{2}{n}\right) \lambda\lambda k_n\left(-y-\frac{2}{n}\right) \overset{(13)}{=} \lambda k_n\left(-x-\frac{2}{n}\right) \lambda k_n\left(-y-\frac{2}{n}\right)$$

$$\overset{(10)}{=} K_{n,\lambda}(x)\, K_{n,\lambda}(y).$$

Thus, we have proved that $K_{n,\lambda}$ satisfies (6). The fact that (7) holds is obvious from (10). This completes the proof of the theorem.

From (10), it is clear that if $\mathscr{E}$ contains at least one element $\lambda$, *different from* $e$, such that $\lambda$ satisfies (11) and (12), then the extension $K_{n,\lambda}$ of $k_n$ is *not unique*.

Theorem 4. *Every group homomorphism* $K_n : (\Delta_n^*, \tau^n) \to \mathscr{E}$, *which is an extension of the subgroup homomorphism* $k_n : (\Delta_n, \tau^n) \to \mathscr{E}$, *is of the form* (10) *with* $\lambda$ *satisfying* (11) *and* (12).

**Proof.** Since $K_n$ is an extension of $k_n$, therefore, we have (7). Now we determine the form of $K_n(x)$ for $x \in \Gamma_n$. Let $x$ be any element of $\Gamma_n$. Then, for all $z \in \Gamma_n$, we have

$$K_n(x) = K_n(x\tau^n z\tau^n z') \overset{(6)}{=} K_n(x\tau^n z)\, K_n(z') \overset{(7)}{=} k_n(x\tau^n z)\, K_n(z')$$

$$\overset{(h)}{=} k_n\left[\left(-x-\frac{2}{n}\right)\tau^n\left(-z-\frac{2}{n}\right)\right] K_n(z') \overset{(5)}{=} k_n\left(-x-\frac{2}{n}\right)$$

$$k_n\left(-z-\frac{2}{n}\right) K_n(z')$$

$$\overset{(6)}{=} k_n\left(-x-\frac{2}{n}\right) K_n\left(-z-\frac{2}{n}\right) K_n(z') = k_n\left(-x-\frac{2}{n}\right)$$

$$K_n\left[\left(-z-\frac{2}{n}\right)\tau^n z'\right].$$

But

$$\left(-z-\frac{2}{n}\right)\tau^n z' = \left(-z-\frac{2}{n}\right)\tau^n\left(\frac{-z}{1+nz}\right) = \left(-z-\frac{2}{n}\right)$$

$$+\left(\frac{-z}{1+nz}\right)+n\left(-z-\frac{2}{n}\right)\left(\frac{-z}{1+nz}\right) = -\frac{2}{n}.$$

Hence

$$K_n(x) = k_n\left(-x-\frac{2}{n}\right)\lambda, \qquad \text{where } \lambda = K_n\left(-\frac{2}{n}\right).$$

Similarly,

$$K_n(x) = K_n(z'\tau^n z\tau^n x) \overset{(6)}{=} K_n(z')\, K_n(z\tau^n x) \overset{(7)}{=} K_n(z')\, k_n(z\tau^n x)$$

$$\overset{(h)}{=} K_n(z')\, k_n\left[\left(-z-\frac{2}{n}\right)\tau^n\left(-x-\frac{2}{n}\right)\right] \overset{(5)}{=} K_n(z')\, k_n\left(-z-\frac{2}{n}\right)$$

$$k_n\left(-x-\frac{2}{n}\right)$$

$$= K_n\left[z'\tau^n\left(-z-\frac{2}{n}\right)\right] k_n\left(-x-\frac{2}{n}\right) = K_n\left(-\frac{2}{n}\right) k_n\left(-x-\frac{2}{n}\right)$$

$$= \lambda k_n\left(-x-\frac{2}{n}\right).$$

Thus

$$K_n(x) = k_n\left(-x-\frac{2}{n}\right)\lambda = \lambda k_n\left(-x-\frac{2}{n}\right), \qquad x \in \Gamma_n,\ \lambda = K_n\left(-\frac{2}{n}\right).$$

But

$$K_n\left[\left(-\frac{2}{n}\right)\tau^n\left(-\frac{2}{n}\right)\right] = K_n\left(-\frac{2}{n}\right)K_n\left(\frac{-2}{n}\right) = \lambda\lambda.$$

Actual computation gives $\left(\frac{-2}{n}\right)\tau^n\left(\frac{-2}{n}\right) = 0$, the identity element of $(\Delta_n^*, \tau^n)$. Since $K_n$ is a group homomorphism, therefore, we must have $K_n(0) = e$, the identity element of $\mathscr{E}$. Thus $\lambda\lambda = e$, which is (11). Since there may exist more than one $\lambda$ satisfying (11), writing $K_n$ as $K_{n,\lambda}$, the required conclusion follows.

**Remark.** Let us define mappings $\phi_n: \Delta_n^* = R - \{0\}$ as

$$\phi_n(x) = nx + 1, \qquad x \in \Delta_n^*.$$

Then, it can be easily verified that

$$\phi_n(x\tau^n y) = \phi_n(x)\,\phi_n(y), \qquad \text{for all } x \text{ and } y \text{ in } \Delta_n^*.$$

Also, $\phi_n(x) > 0$ if and only if $x \in \Delta_n$. Since $\phi_n$ also is a bijection, therefore, from the above observations, it follows that $\phi_n$ induces an isomorphism between $(\Delta_n, \tau^n)$ and the group $(\nabla_0, \cdot)$ where $\nabla_0 = (0, \infty)$. Then, (5) can be written in the form

$$k_n(\phi_n^{-1}(uv)) = k_n(\phi_n^{-1}(u))\;k_n(\phi_n^{-1}(v)), \qquad u > 0,\ v > 0.$$

If we write

$$\psi_n(u) = k_n(\phi_n^{-1}(u)), \qquad u > 0,$$

then

(B) $$\psi_n(uv) = \psi_n(u)\,\psi_n(v), \qquad u > 0,\ v > 0.$$

The main advantage in dealing with (B) is that the argument of $\psi_n$ on the L.H.S. of (B) is also independant of $n$ as compared with that of $k_n$ in (5). If $K_n$ is an extension of $k_n$, then by the above reasoning, the mapping $\Psi_n$: $\mathbf{R} - \{0\} \to \mathscr{E}$ defined by

$$\Psi_n(u) = K_n(\phi_n^{-1}(u)), \qquad u \neq 0,$$

is a homomorphism of $R - \{0\}$ into $\mathscr{E}$ and it extends $\psi_n$. If we can find the extension $\Psi_n$, then with the aid of $\psi_n$, we can also find the corresponding form of $K_n$. However, the method explained in the proof of theorem 4 readily gives us the forms of extensions if they exist and theorem 3 ensures that they are indeed the extensions of $k_n$.

Now we give an extension theorem concerning the functional equation (3).

Theorem 5. *If the functions $f_n$, $g_n$, $h_n$ defined on the subgroup $(\Delta_n, \tau^n)$ satisfy the functional equation* (3) *with their values lying in a group $\mathscr{E}$ containing no zero element, then the functions $F_n$, $G_n$, $H_n$ defined on the group $(\Delta_n^*, \tau^n)$, with their values in $\mathscr{E}$, by*

$$(14) \qquad F_n(x) = g_n(0)\, K_n(x)\, h_n(0), \;\; G_n(x) = g_n(0)\, K_n(x), \;\; H_n(x) = K_n(x)\, h_n(0),$$

*where $K_n$ is an extension of the subgroup homomorphism $k_n : (\Delta_n, \tau^n) \to \mathscr{E}$ satisfying* (5), *are the extensions of $f_n$, $g_n$, $h_n$ respectively in the sense that they satisfy* (8) *and* (9).

Proof. We have

$$F_n(x\tau^n y) \overset{(14)}{=} g_n(0)\, K_n(x\tau^n y)\, h_n(0) = g_n(0)\, K_n(x)\, K_n(y)\, h_n(0) \overset{(14)}{=} G_n(x)\, H_n(y).$$

This proves the theorem.

In theorem 5, we have assumed that $\mathscr{E}$ contains no zero element. If $\mathscr{E}$ contains a zero element, say $b$, then

$$f_n(x) = b, \qquad g_n(x) \text{ arbitrary}, \qquad h_n(x) = b,$$

$$f_n(x) = b, \qquad g_n(x) = b, \qquad h_n(x) \text{ arbitrary},$$

are also (trivial) solutions of (3). The extensions of these solutions are not of any importance and we shall not consider them.

For a fixed $\lambda$, let us write $F_n$, $G_n$ and $H_n$ as $F_{n,\lambda}$, $G_{n,\lambda}$ and $H_{n,\lambda}$ respectively. Then (10) and (14) give

$$(15)\quad \begin{cases} F_{n,\lambda}(x) = g_n(0)\, k_n(x)\, h_n(0), & x \in \Delta_n, \\ \qquad\quad = g_n(0)\, \lambda k_n\left(-x - \dfrac{2}{n}\right) h_n(0), & x \in \Gamma_n, \\ G_{n,\lambda}(x) = g_n(0)\, k_n(x), & x \in \Delta_n, \\ \qquad\quad = g_n(0)\, \lambda k_n\left(-x - \dfrac{2}{n}\right), & x \in \Gamma_n, \\ H_{n,\lambda}(x) = k_n(x)\, h_n(0), & x \in \Delta_n, \\ \qquad\quad = \lambda k_n\left(-x - \dfrac{2}{n}\right) h_n(0), & x \in \Gamma_n. \end{cases}$$

where $\lambda$ satisfies (11) and (12). Also from (4), we have

$$(16)\quad k_n(x) = [g_n(0)]' f_n(x)\, [h_n(0)]' = [g_n(0)]'\, g_n(x) = h_n(x)\, [h_n(0)]', \quad x \in (\Delta_n; \tau^n).$$

Hence, in terms of $g_n$, (12) and (13) reduce to the form

$$(17)\qquad \lambda\, [g_n(0)]'\, g_n(x) = [g_n(0)]'\, g_n(x)\, \lambda, \qquad x \in \Delta_n,$$

and

$$(18)\qquad \lambda [g_n(0)]' g_n\left(-x - \frac{2}{n}\right) = [g_n(0)]'\, g_n\left(-x - \frac{2}{n}\right) \lambda, \qquad x \in \Gamma_n.$$

Similarly, in terms of $h_n$, (12) and (13) take the form

$$(19)\qquad \lambda h_n(x)\, [h_n(0)]' = h_n(x)\, [h_n(0)]'\, \lambda, \qquad x \in \Delta_n,$$

and

$$(20)\qquad \lambda h_n\left(-x - \frac{2}{n}\right) [h_n(0)]' = h_n\left(-x - \frac{2}{n}\right) [h_n(0)]'\, \lambda, \qquad x \in \Gamma_n.$$

Also (15) reduces to

$$(21)\quad \begin{cases} F_{n,\lambda}(x) = f_n(x), & x \in \Delta_n, \\ \qquad = g_n(0)\,\lambda\,[g_n(0)]'\, f_n\left(-x - \dfrac{2}{n}\right), & x \in \Gamma_n, \\ G_{n,\lambda}(x) = g_n(x), & x \in \Delta_n, \\ \qquad = g_n(0)\,\lambda\,[g_n(0)]'\, g_n\left(-x - \dfrac{2}{n}\right), & x \in \Gamma_n, \\ H_{n,\lambda}(x) = h_n(x), & x \in \Delta_n, \\ \qquad = \lambda h_n\left(-x - \dfrac{2}{n}\right), & x \in \Gamma_n. \end{cases}$$

Now, we prove the following theorem :

Theorem **6.** *If the functions $f_n, g_n, h_n$ defined on the subgroup $(\Delta_n, \tau^n)$, satisfy the functional equation* (3) *with their values lying in a group $\mathscr{E}$ containing no zero element, then for each $\lambda \in \mathscr{E}$ satisfying* (11), *the functions $F_{n,\lambda}$, $G_{n,\lambda}$ and $H_{n,\lambda}$ defined by* (21) *are extensions of $f_n$, $g_n$ and $h_n$ respectively in the sense that they satisfy* (8) *and* (9).

**Proof.** As in the proof of theorem 3, we discuss the same four cases.

*Case (i).* $x \in (\Delta_n, \tau^n)$, $y \in (\Delta_n, \tau^n)$. Then,

$$F_{n,\lambda}(x\tau^n y) = f_n(x\tau^n y) = g_n(x)\, h_n(y) = G_{n,\lambda}(x)\; H_{n,\lambda}(y).$$

*Case (ii).* $x \in (\Delta_n, \tau^n)$, $y \in \Gamma_n$. Then

$$F_{n,\lambda}(x\tau^n y) \overset{(21)}{=} g_n(0)\,\lambda\,[g_n(0)]'\, f_n\left[-(x\tau^n y) - \frac{2}{n}\right]$$

$$= g_n(0)\,\lambda\,[g_n(0)]'\, f_n\left[x\tau^n\left(-y - \frac{2}{n}\right)\right]$$

$$\overset{(3)}{=} g_n(0)\,\lambda\,[g_n(0)]'\, g_n(x)\, h_n\left(-y - \frac{2}{n}\right)$$

$$\overset{(17)}{=} g_n(0)\,[g_n(0)]'\,g_n(x)\,\lambda h_n\left(-y-\frac{2}{n}\right)$$

$$= g_n(x)\,\lambda h_n\left(-y-\frac{2}{n}\right) \overset{(21)}{=} G_{n,\lambda}(x)\; H_{n,\lambda}(y).$$

*Case (iii).* $x\in\Gamma_n,\ \ y\in(\Delta_n,\tau^n)$. Then

$$F_{n,\lambda}(x\tau^n y) \overset{(21)}{=} g_n(0)\,\lambda\,[g_n(0)]'\,f_n\left[-(x\tau^n y)-\frac{2}{n}\right]$$

$$= g_n(0)\,\lambda\,[g_n(0)]'\,f_n\left[\left(-x-\frac{2}{n}\right)\tau^n y\right]$$

$$\overset{(3)}{=} g_n(0)\,\lambda\,[g_n(0)]'\,g_n\left(-x-\frac{2}{n}\right)h_n(y)$$

$$\overset{(21)}{=} G_{n,\lambda}(x)\; H_{n,\lambda}(y).$$

*Case (iv).* $x\in\Gamma_n,\ \ y\in\Gamma_n$. Then,

$$F_{n,\lambda}(x\tau^n y) = f_n(x\tau^n y) = f_n\left[\left(-x-\frac{2}{n}\right)\tau^n\left(-y-\frac{2}{n}\right)\right]$$

$$\overset{(3)}{=} g_n\left(-x-\frac{2}{n}\right)h_n\left(-y-\frac{2}{n}\right)$$

$$= g_n(0)\;[g_n(0)]'\,g_n\left(-x-\frac{2}{n}\right)\lambda\lambda h_n\left(-y-\frac{2}{n}\right)$$

$$\overset{(18)}{=} g_n(0)\;\lambda\,[g_n(0)]'\;g_n\left(-x-\frac{2}{n}\right)\lambda h_n\left(-y-\frac{2}{n}\right)$$

$$= G_{n,\lambda}(x)\;H_{n,\lambda}(y).$$

This completes the proof of the theorem.

We hope to discuss some more methods of extending subgroup homomorphisms elsewhere.

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## ÖZET


ACZÉL, $x'$, grup işlemine göre $x$ elementinin tersini göstermek üzere,

$$x \in G \Rightarrow x \in S \text{ veya } x' \in S$$

koşulunu sağalayan $G$ grubunun bir $S$ alt-semigrubu üzerinde tanımlanan ve bu $S$ semigrubunu bir $H$ grubunun içine tasvir eden bir $f$ homomorfizmasının, her $x, y \in G$ için

$$g(xy) = g(x)g(y) \tag{f}$$

olacak şekilde $G$ grubundan $H$ grubuna bir $g$ homomorfizmasına $f$ ve $g$ $S$ üzerinde çakışacak tarzda tek bir şekilde genişletebileceğini göstermiştir.

Bu araştırmada ise $S$ nin, $G$ grubunun bir alt grubu olması halinde yukarıdakine benzer teoremler elde etmeğe uğraşdmaktadır: $S$ nin bir semigrup değil, bir grup olması ACZÉL'in yönteminin bu hale uygulanmasını önlemektedir. Ayrıca (f) fonksiyonel denklemine bâzı başka şekiller verilmesi de öngörülmüştür.