ON WAVE SOLUTIONS OF WEAKENED FIELD EQUATIONS IN A $V_2 \times V_2$ SPACE-TIME

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KILMISTER and NEWMANN [1] [1)] and LOVELOCK [4] have mentioned field equations as alternative to the vacuum field equations of the EINSTEIN theory of general relativity. In this paper we have considered these equations in a Riemannian fourfold of class two representing the product of two surfaces ie.. $V_2 \times V_2$ space-time and it is found that the wave solution exist.

1. Introduction. KILMISTER and NEWMANN [1] proposed an alternative set of field equations in general relativity which, in the absence of sources, are given by

$$R^h_{ijk;h} = 0 \tag{1.1}$$

where a semi-colon (;) denotes covariant differentiation with respect to CHRISTOFFEL symbol $\{^{k}_{ij}\}$. The space which are interpreted as the gravitational field in vacuo in the orthodox theory of general relativity form only a subset of such spaces for these field equations. The field equations (1.1) are called «weakened field equations» (ie. weaker than the EINSTEIN equations of general relativity in vacuo) in the sense that each of them admits a class of solutions for which

$$R_{ij} = 0, \tag{1.2}$$

as a sub-class of solution.

1) Numbers in brackets refer to the references at the end of the paper.

On contracting the BIANCHI identities which hold in a general Riemannian space and using the result in (1.1), it is easy to see that the form (1.1) of the weakened field equations is equivalent to

$$R_{ij,k} - R_{ik,j} = 0. \tag{1.3}$$

Although the physical implications of the weakened field equations are yet not well-established but many others have tried to find the solutions of these field equations in the hope that these field equations may he useful in future. THOMPSON [2] made investigations of the weakened field equations and found several different static, spherically symmetric solutions not transformable into one another and showed conclusively that the weakened field equations are too weak. KILMISTER [3] has surveyed the question of alternative field equations in general relativity. Further, LOVELOCK [4], [5] has solved a number of alternative set of weakened field equations including (1.1) and has obtained a static spherically symmetric solution which represents the field of a massless charged particle at rest at the origin for all time. SWAMI [6] has found three solutions of the weakened field equations (1.3) with $R_{ij} \neq 0$, $R_{ij} \neq \lambda g_{ij}$ and has discussed some of the geometrical and dynamical aspect of these solutions. Recently LAL and SINGH [7] have found the solutions of the field equations (1.3) with some useful conclusion, in the cylindrical symmetric space-time with metric [8]. LOVELOCK [4] has also mentioned many other field equations as alternatives to the vacuum field equations of the EINSTEIN theory of general relativity. One of these field equations is

$$H_k^{ij} \equiv R^{ij}_{;k} = 0. \tag{1.4}$$

In the present paper we have considered the field equations (1.3) and (1.4) in a $V_2 \times V_2$ space-time with metric [8]:

$$ds^2 = - A(dx^2 + dy^2) - B(dz^2 - dt^2), \tag{1.5}$$

where $A = A(x, y)$, $B = B(z, t)$ and x, y, z, t correspond to x^1, x^2, x^3, x^4 respectively.

2. Solution of the field equations (1.3). The component g^{ij} corresponding to the metric (1.5) are

$$\text{(2.1)} \qquad \begin{cases} g^{11} = -g^{22} = -1/A, \\ g^{33} = -g^{44} = -1/B \end{cases}$$

and the non-vanishing components of CHRISTOFFEL symbols of second kind $\{^{k}_{ij}\}$ are

$$\text{(2.2)} \qquad \begin{cases} \left\{ {1 \atop 11} \right\} = \left\{ {2 \atop 12} \right\} = -\left\{ {1 \atop 22} \right\} = A_1/2A, \\ \left\{ {2 \atop 22} \right\} = \left\{ {1 \atop 12} \right\} = -\left\{ {2 \atop 11} \right\} = A_2/2A, \\ \left\{ {3 \atop 33} \right\} = \left\{ {4 \atop 34} \right\} = \left\{ {3 \atop 44} \right\} = B_3/2B, \\ \left\{ {4 \atop 44} \right\} = \left\{ {3 \atop 34} \right\} = \left\{ {4 \atop 33} \right\} = B_4/2B. \end{cases}$$

Here the lower suffixes 1, 2, 3, 4 after a function indicate ordinary partial differentiation with respect to x, y, z, t respectively. The RICCI tensor R_{ij} is defined by

$$\text{(2.3)} \qquad R_{ij} = \{^{s}_{is}\}_{,j} - \{^{s}_{ij}\}_{,s} + \{^{t}_{is}\}\{^{s}_{tj}\} - \{^{s}_{ts}\}\{^{t}_{ij}\}\,.$$

Using (2.2) in (2.3) the non-vanishing components of R_{ij} are

$$\text{(2.4)} \qquad \begin{cases} R_{11} = R_{22} = \{(A_{11} + A_{22})\,A - (A_1^2 + A_2^2)\}/2A^2 = X/2, \\ B_{33} = -B_{44} = \{(B_{33} - B_{44})\,B - (B_3^2 - B_4^2)\}/2B^2 = Y/2, \end{cases}$$

where

$$\text{(2.5)} \qquad \begin{cases} X = (A_{11} + A_{22})/A - (A_1^2 + A_2^2)/A^2, \\ Y = (B_{33} - B_{44})/B - (B_3^2 - B_4^2)/B^2. \end{cases}$$

From (2.1) and (2.4) the scalar curvature $R = g^{ij} B_{ij}$ is given by

(2.6) $R = -(A_{11} + A_{22})/A^2 + (A_1^2 + A_2^2)/A^2 + (B_{44} - B_{33})/B^2 + (B_3^2 - B_4^2)/B^3.$

A first contraction of vacuum field equations (1.3) gives

$$B^i_{j,i} - R_{,j} = 0 \tag{2.7}$$

and the twice contracted BIANCHI identities imply

$$R^i_{,j} - (1/2)\, R_{,j} = 0 \tag{2.8}$$

where a comma denotes partial differentiation. From the two equations (2.7) and (2.8) we have $R_{ij} = 0$ which imply that $R =$ constant.

Using the values of $\{^{\,k}_{ij}\}$ from (2.2) in (1.3), we get

(2.9)
a) $A\, \partial_2\, R_{11} - A_2\, R_{11} = 0,$

b) $A\, \partial_1\, B_{22} - A_1\, R_{22} = 0,$

c) $B\, \partial_4\, B_{33} - B_4\, R_{33} = 0,$

d) $B\, \partial_3\, R_{44} - B_3\, R_{44} = 0.$

Using the components of R_{ij} from (2.4) in (2.9) *a* - (2.9) *d*, we get

$$X = K_1\, A, \tag{2.10}$$

$$Y = K_2\, B, \tag{2.11}$$

where K_1 and K_2 are arbitrary constants.

Putting $a = \log A$ and $b = \log B$ in equation (2.10) and (2.11), we get

$$a_{11} + a_{22} = K_1\, e^a, \tag{2.12}$$

$$b_{33} - b_{44} = K_2\, e^b. \tag{2.13}$$

Reducing equation (2.12) to canonical form by changing the dependent variable a into ρ, where $\rho = \rho(\xi, \eta)$ and

$$\xi = x + iy \quad , \quad \eta = x - iy, \tag{2.14}$$

we get (2.12) as

$$\partial^2 \rho / \partial \xi \, \partial \eta = (K_1/4)\, e^a. \tag{2.15}$$

Equation (2.15) is of LIOUVILLE's form and by FORSYTH [9] has a solution of the form

$$e^\rho = 2 f_1'(\xi)\, f_2'(\eta) / [f_1(\xi) + (K_1/4)\, f_2(\eta)]^2, \tag{2.16}$$

where each f_1 and f_2 is an arbitrary function of its argument and prime denotes partial differentiation with respect to it.

Hence exact solution of equation (2.12) is

$$a = \log A = \log [2 f_1'(x + iy)] + \log [f_2'(x - iy)] - 2 \log [f_1(x + iy) + (K_1/4)\, \{f_2(x - iy)\}]. \tag{2.17}$$

Similarly, (2.13) can be reduced to canonical form by changing the dependent variable b into ρ_1, where $\rho_1 = \rho_1(\xi_1, \eta_1)$ and

$$\xi_1 = z + t \quad , \quad \eta_1 = z - t, \tag{2.18}$$

we get (2.13) as

$$\partial^2 \rho_1 / \partial \xi_1 \, \partial \eta_1 = (K_2/4)\, e^b, \tag{2.19}$$

which gives the exact solution of equation (2.13) in the form

$$b = \log B = \log [2 g_1'(z + t)] + \log [g_2'(z - t)] - 2 \log [g_1(z + t) + (K_2/4)\, \{g_2(z - t)\}], \tag{2.20}$$

where each g_1 and g_2 is an arbitrary function of its argument and prime denotes partial differentiation with respect to it.

Now using (2.10) and (2.11) in (2.6), we get

(2.21) $$R = -(K_1 + K_2) = K$$

where K is another constant, which is consistent with the result that in a RIEMANNIAN space-time where weakened field equations hold R must he constant.

Thus, equations (1.5), (2.17) and (2.20) (for which $R_{ij} \neq 0$ and $R_{ij} \neq \lambda g_{ij}$, λ being a constant) constitute wave solutions of the weakened field equation (1.3) in a $V_2 \times V_2$ space-time.

3. Solutions of equation (1.4). From (2.1) and (2.4) the non-vanishing components of the tensor R^{ij} are

(3.1) $$\begin{cases} R^{11} = R^{22} = X/2A^2 \\ R^{33} = -R^{44} = Y/2B^2. \end{cases}$$

Using the components of R^{ij} from (3.1) in the field equations (1.4), we get

(3.2)
$$\text{a)}\quad \partial_1 X - A_1 X/A = 0,$$
$$\text{b)}\quad \partial_2 X - A_2 X/A = 0,$$
$$\text{c)}\quad \partial_3 Y - B_3 Y/B = 0,$$
$$\text{d)}\quad \partial_4 Y - B_4 Y/B = 0.$$

From (3.2) a) - (3.2) d), we have

(3.3) $$X = k_3 A,$$

(3.4) $$Y = k_4 B,$$

where k_3 and k_4 are arbitrary constants.

Putting again $a = \log A$ and $b = \log B$ in equation (3.3) and (3.4), we get

$$a_{11} + a_{22} = k_3 e^a, \tag{3.5}$$

$$b_{33} - b_{44} = k_4 e^b. \tag{3.6}$$

The exact solutions of (3.5) and (3.6) are respectively in the form

$$\begin{aligned} a = \log A &= \log [2F_1'(x + iy)] + \log [F_2'(x - iy)] \\ &\quad - 2 \log [F_1(x + iy) + (k_3/4) \{F_2(x - iy)\}], \end{aligned} \tag{3.7}$$

$$\begin{aligned} b = \log B &= \log [2G_1'(z + t)] + \log [G_2'(z - t)] \\ &\quad - 2 \log [G_1(z + t) + (k_4/4) \{G_2(z - t)\}], \end{aligned} \tag{3.8}$$

where F_1, F_2 and G_1, G_2 are arbitrary functions of their arguments and primes denote the partial differentiation with respect to their arguments.

Thus equations (1.5), (3.7), (3.8) constitute wave solutions of the weakened field equations (1.4).

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ÖZET

KILMISTER ve NEWMANN [1] ve LOVELOCK [4], EINSTEIN'in genel relativite teorisinde geçen boşluktaki alan denklemlerinin yerini alabilecek alan denklemleri öne sürmüşlerdir. Bu çalışmada ikinci sınıftan dört boyutlu bir RIEMANN uzayı olarak düşünülen iki yüzeyin çarpımı olarak elde edilmiş bir uzay-zaman evreni, yani $V_2 \times V_2$ biçiminde bir evrende bu denklemler incelenmiş ve dalga çözümlerinin varlığı kanıtlanmıştır.