BIANCHI IDENTITIES AND THE PROJECTIVELY FLAT SPECIAL KAWAGUCHI SPACE

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The purpose of the present paper is to find certain identities regarding some conformal entity and to study their properties in a projectively flat space.

1. Introduction. The theory of n - dimensional special Kawaguchi space $K_n^{(1)}$ in which the arc length of a curve $x^i = x^i(t)$ is given by the integral

$$s = \int (A_i \, x''^i + B) \, dt,$$

was developed by A. KAWAGUCHI [1] [2)]. Here A_i and B are differentiable homogeneous functions of degrees $p - 2$ and p respectively. A. KAWAGUCHI has defined a connection in $K_n^{(1)}$ by introducing «Craig Vector» of the function $F(x, x', x'')$. It is given by

$$(1.1) \qquad T_i \overset{\text{def}}{=} (A_{k(i)} - A_{i(k)}) \, x''^k - 2A_{ik} \, x'^k + B_{(i)} \, ,$$

where

$$A_{k(i)} = \partial'_i A_k \, , \quad A_{ik} = \partial_k A_i \, , \quad B_{(i)} = \partial'_i B.$$

Here ∂'_i and ∂_i denotes $\partial/\partial x'^i$ and $\partial/\partial x^i$ respectively.

If $p \neq 3/2$, we have

$$(1.2) \qquad x^{(2)i} = x''^i + 2\Gamma^i,$$

1) Communicated by Prof. (Dr.) Ram Behari.

2) Numbers in brackets refer to the references at the end of the paper.

where

$$(1.3)\qquad \begin{cases} 2\Gamma^i_{\cdot} = (2A_{lk}\, x'^k - B_{(l)})\, G^{li}, \\ G_{ik} = 2A_{i(k)} - A_{k(i)}\, , \\ G_{ik}\, G^{il} = \delta^l_k\, . \end{cases}$$

Let X^i he a contravariant vector field homogeneous of degree zero with respect to x'^i, then the covariant derivative of X^i is given by

$$(1.4)\qquad \nabla_j X^i = \partial_j X^i - \partial'_k X^i\, \Gamma^k_{(j)} + X^k\, \Gamma^i_{(k)(j)}\, ,$$

where $\Gamma^i(x, x')$ is the connection parameter, positively homogeneous of degree two with regard to directional argument. We have the following relations satisfied by the curvature tensors of the special Kawaguchi space:

$$(1.5)\qquad \text{a)}\ R_{jkl}^{\cdots i} = -R_{kjl}^{\cdots i}\, , \qquad \text{b)}\ B_{[jkl]}^{\cdots i} = 0,$$

$$\text{c)}\ K_{jk}^{\cdot\cdot i} = R_{jkl}^{\cdots i}\, x'^l\, , \qquad \text{d)}\ R_{jkl}^{\cdots i} = K_{jk(l)}^{\cdots i} = \nabla'_l\, K_{jk}^{\cdot\cdot i}\, ,$$

$$\text{e)}\ B_{jkl}^{\cdots i} = \Gamma^i_{(j)(k)(l)}\, , \qquad \text{f)}\ B^i_{jkl}\, x'^l = 0.$$

The identities of Bianchi are expressed by

$$(1.6)\qquad \text{a)}\ \nabla_{[h}\, R_{jk]l}^{\cdots i} + K_{[hj}^{\cdot\cdot r}\, B_{k]lr}^{\cdots i} = 0,$$

$$\text{b)}\ 2\nabla_{[h}\, B_{jk]l}^{\cdots i} + \nabla'_l\, R_{hjk}^{\cdots i} = 0,$$

$$\text{c)}\ \nabla_{[h}\, K_{jk]}^{\cdot\cdot i} = 0.$$

The projective deviation tensor $W^i_j(x, x')$ in $K^{(1)}_n$ is given by

$$(1.7)\qquad W^i_j = H^i_j - H\,\delta^i_j - \frac{x'^i}{n+1}\,(\partial'_a\, H^a_j - \partial'_j\, H),$$

where

$$(1.8)\qquad H^i_k = K_{jk}^{\cdot\cdot i}\, x'^j \quad \text{and} \quad H = \frac{1}{n-1}\, H^i_i\, .$$

2. Fundamentals of conformai transformation in $K_n^{(1)}$. The conformai transformation $K_n^{(1)}$ has been considered by C. KANO [4], in which the connection $\bar{\Gamma}^i$ is defined by

$$\bar{\Gamma}^i = \Gamma^i + \alpha x'^i, \tag{2.1}$$

where

$$\alpha = \frac{1}{2p-3}(\sigma_j x'^j - 2\Gamma^j \sigma_{(j)}) = \frac{1}{2p-3} x'^j \nabla_j \sigma. \tag{2.2}$$

Here $\alpha(x, x')$ and $\sigma(x, x')$ are homogeneous functions of degree one and zero respectively with respect to x'^i. In view of this transformation the two metric functions F and $\bar{F}$ related by $\bar{F} = \sigma(x, x') F$ satisfy the Zermelo's condition. M. OKUMURA [5] has defined a connection in the following way :

$$\pi^i_{jk} = \Gamma^i_{(j)(k)} - \frac{1}{n+1} \Gamma^r_{(r)(j)(k)} x'^i. \tag{2.3}$$

3. BIANCHI identities in $K_n^{(1)}$. Differentiating (2.1) partially with respect to x'^j and x'^k in successive order and using the homogeneity poperty of α, we obtain

(3.1) a) $\alpha = \dfrac{1}{n+1}(\bar{\Gamma}^i_{(i)} - \Gamma^i_{(i)})$,

b) $\alpha_{(j)} = \dfrac{1}{n+1}(\bar{\Gamma}^i_{(i)(j)} - \Gamma^i_{(i)(j)})$,

so that

c) $\alpha_{(j)(k)} = \dfrac{1}{n+1}(\bar{\Gamma}^i_{(i)(j)(k)} - \Gamma^i_{(i)(j)(k)})$.

From (2.1), (2.3), (3.1a), (3.1b) and (3.1c), we get

$$\tilde{H}^i_{jk} \overset{\text{def}}{=} \pi^i_{jk} - \frac{1}{n+1}\Gamma^r_{(r)(j)}\delta^i_k - \frac{1}{n+1}\Gamma^r_{(r)(k)}\delta^i_j\,. \tag{3.2}$$

These entities are invariant under the conformal change and will he called as the coefficients of conformai connection. $\tilde{H}^i_{jk}(x, x')$ is symmetric in its lower indices and is homogeneous of degree zero with respect to x'^i. We define the conformai covariant derivative of a vector X^i with respect to x^j for connection parameter $\tilde{H}^i_{jk}$ in the following way:

$$(3.3) \qquad \overline{\nabla}_j X^i = \partial_j X^i - \partial'_k X^i \tilde{H}^k_{jr} x'^r + X^k \tilde{H}^i_{kj}.$$

We have the operators ∇, ∇^* and $\overline{\nabla}$ denoting the covariant derivative with respect to connections $\Gamma^i_{(j)(k)}$, π^i_{jk} and $\tilde{H}^i_{jk}$, respectively. In view of (3.1), we have

$$(3.4) \qquad (\overline{\nabla}_j \overline{\nabla}_k - \overline{\nabla}_k \overline{\nabla}_j) X^i = - Z^i_{jkh} X^h + \partial'_h X^i Z^h_{jkp} x'^p,$$

where $Z^i_{jkh}(x, x')$ is homogeneous function of degree zero with respect to x'^i and is skew symmetric with respect to indices j and k. Using relations (1.4), (2.3), (3.2) and (3.3), we get

$$(3.5) \qquad (\overline{\nabla}_j - \nabla^*_j) X^i = \frac{1}{n+1} (\Gamma^r_{(r)(j)} x'^k + \Gamma^r_{(r)} \delta^k_j) \partial'_k X^i$$

$$- \frac{1}{n+1} (\Gamma^r_{(r)(j)} \delta^i_k + \Gamma^r_{(r)(k)} \delta^i_j) X^k$$

and

$$(3.6) \qquad (\nabla^*_j - \nabla_j) X^i = - \frac{1}{n+1} \Gamma^h_{(h)(j)(k)} x'^i X^k.$$

M. OKUMURA [5] has defined a second kind of conformal curvature tensor D^{*i}_{jkh}, given by

$$(3.7) \qquad D^{*i}_{jkh} = \tilde{H}^i_{jk(h)}.$$

Let us suppose a covariant vector $\eta_i(x^k)$ which is only a function of positional coordinate x^k. Using the commutation formula (3.4) for η_i, we get

$$(3.8)\qquad (\overline{\nabla}_j\,\overline{\nabla}_k - \overline{\nabla}_k\,\overline{\nabla}_j)\,\eta_i = Z^h_{jki}\,\eta_h\,.$$

Differentiating (2.1) covariantly with respect to x^m and commutating the obtained result with respect to indices m, j, k and then adding all the three equations thus obtained, we get

$$(3.9)\qquad (\overline{\nabla}_{[m}\,\overline{\nabla}_j - \overline{\nabla}_{[j}\,\overline{\nabla}_m)\,\overline{\nabla}_{k]}\,\eta_i = \overline{\nabla}_{[m}\,Z^h_{jk]i}\,\eta_h + \overline{\nabla}_{[m}\,\eta_{<h>}\,Z^h_{jk]i}\,.$$

Using the commutation formulae (3.4) and the relation (3.7), we obtain

$$(3.10)\qquad \overline{\nabla}_{[m}\,Z^h_{jk]i} + D^{*h}_{ir[k}\,Z^r_{mj]p}\,x'^p = 0,$$

where

$$(3.11)\qquad \text{a)}\quad Z^h_{[jki]} = 0 \quad\text{and}\quad \text{b)}\quad \dot\partial_r\,\overline{\nabla}_m\,\eta_i = -\,\eta_p\,\dot\partial_r\,\widetilde{H}^p_{mi}\,.$$

Thus, we have

Theorem (3.1). *The Bianchi identity for the conformal entity $Z^i_{jkh}(x, x')$ in $K^{(1)}_n$ is given by* (3.10).

Applying (3.5) for Z^i_{jkh}, we get

$$(3.12)\qquad (\overline{\nabla}_m - \nabla^*_m)\,Z^i_{jkh} = \frac{1}{n+1}\,\dot\partial_r\,Z^i_{jkh}\,(\Gamma^s_{(s)}\,\delta^m_r + \Gamma^s_{(s)(m)}\,x'^r)$$
$$-\frac{1}{n+1}\,[\Gamma^s_{(s)(m)}\,\delta^i_r + \Gamma^s_{(s)(r)}\,\delta^i_m]\,Z^r_{jkh} +$$
$$+\frac{1}{n+1}\,[\Gamma^s_{(s)(m)}\,\delta^r_j + \Gamma^s_{(s)(j)}\,\delta^r_m]\,Z^i_{rkh} +$$
$$+\frac{1}{n+1}\,[\Gamma^s_{(s)(m)}\,\delta^r_k + \Gamma^s_{(s)(k)}\,\delta^r_m]\,Z^i_{jrh} +$$
$$+\frac{1}{n+1}\,[\Gamma^s_{(s)(m)}\,\delta^r_h + \Gamma^s_{(s)(h)}\,\delta^r_m]\,Z^i_{jkr}\,.$$

Commutating (3.12) cyclically with respect to indices m, j, k and adding all the three equations thus obtained and using (3.11) a, (3.10), we get

$$(3.13)\qquad \overset{*}{\nabla}_{[m} Z^i_{jk]h} = \frac{1}{n+1}\{\delta^i_{[m} Z^r_{jk]h} \Gamma^s_{(s)(r)} - \partial'_{[m} Z^i_{jk]h} \Gamma^s_{(s)}\} - D^{*i}_{hr[k} Z^r_{mj]p} x'^p.$$

Thus, we have

Theorem (3.2). *In a* $K_n^{(1)}$ *the Bianchi identity for* $Z^i_{jkh}(x, x')$ *is given by the equation* (3.13).

Similarly, using the commutation formulae (3.6), we get the following Bianchi identity:

Theorem (3.3). *The Bianchi identity for* $Z^i_{jkh}(x, x')$ *in* $K_n^{(1)}$ *has the form*

$$\nabla_{[m} Z^i_{jk]h} = \frac{1}{n+1}\{\delta^i_{[m} Z^r_{jk]h} \Gamma^s_{(s)(r)} + \Gamma^r_{(r)(s)[(m)} Z^s_{jk]h} x'^i - \partial'_{[m} Z^i_{jk]h} \Gamma^s_{(s)} - \Gamma^r_{(r)(h)[(m)} Z^i_{jk]s} x'^s\} - D^{*i}_{hr[k} Z^r_{mj]p} x'^p.$$

4. Projectively flat $K_n^{(1)}$. In an n-dimensional special Kawagnchi space S. KAWAGUCHI [2] has considered a projective transformation in which the function $'\Gamma^i(x, x')$ is given by

$$(4.1)\qquad '\Gamma^i = \Gamma^i + \hat{\alpha}(x, x')\, x'^i.$$

Here $\hat{\alpha}(x, x')$ is a scalar function homogeneous of degree one. By virtue of (4.1) the curvature tensor $K_{jk}^{\cdots i}$ reduces to the form

$$(4.2)\qquad 'K_{jk}^{\cdots i} = K_{jk}^{\cdots i} + x'^i(\partial'_j \psi_k - \partial'_k \psi_j) + \delta^i_j \psi_k - \delta^i_k \psi_j ,$$

where

$$(4.3)\qquad \psi_k = \nabla_k \hat{\alpha} - \hat{\alpha} \cdot \hat{\alpha}_{(k)} .$$

$\psi_k(x, x')$ is homogeneous function of degree one with respect to x'^i. Differentiating (4.3) partially with respect to x'^j and commutating the obtained relation, we get

$$(4.4) \qquad \nabla'_j \psi_k - \nabla'_k \psi_j = \nabla_k \hat{\alpha}_{(j)} - \nabla_j \hat{\alpha}_{(k)},$$

where

$$(\nabla_j \nabla'_k - \nabla'_k \nabla_j) \hat{\alpha} = 0.$$

Thus, in order that if it be possible to find such a projective change (4.1) for which $'K_{jk}^{\cdots i}$ vanishes, then there must exist a vector field $\psi_k(x, x')$ satisfying the relation

$$(4.5) \qquad K_{jk}^{\cdots i} = \delta_k^i \psi_j - \delta_j^i \psi_k - x'^i(\partial'_j \psi_k - \partial'_k \psi_j).$$

Differentiating (4.3) and using equations (4.4) and (4.5), we get the following integrability condition

$$(4.6) \qquad \nabla_j \psi_k - \nabla_k \psi_j = 0.$$

From (3.5), we have

$$(4.7) \qquad \psi_k = \frac{n}{n^2-1} K_{ki}^{\cdots i} + \frac{1}{n^2-1} R_{ijk}^{\cdots i} x'^j.$$

By virtue of (4.6) and (4.7), we obtain

$$(4.8) \qquad (\nabla_h R_{ijk}^{\cdots i} - \nabla_k R_{ijh}^{\cdots i}) x'^j + n(\nabla_h K_{ki}^{\cdots i} - \nabla_k K_{hi}^{\cdots i}) = 0.$$

Thus, we have

Theorem (4.1). *In a* $K_n^{(1)}$ *the integrability condition* (4.6) *reduces to* (4.8) *if and only if there exists a projective change for which* $'K_{jk}^{\cdots i} = 0$.

The projective curvature tensor $W_{jk}^i(x, x')$ [2] is given by

$$(4.9) \qquad W_{jk}^i = K_{jk}^{\cdots i} + \frac{x'^i}{n+1}(\partial'_j K_{ka}^{\cdots a} - \partial'_k K_{ja}^{\cdots a}) + $$
$$+ \frac{\delta_j^i}{n+1}\left\{ K_{ka}^{\cdots a} + \frac{1}{n-1}\partial'_k(K_{ab}^{\cdots a} x'^b)\right\} - $$
$$- \frac{\delta_k^i}{n+1}\left\{ K_{ja}^{\cdots a} + \frac{1}{n-1}\partial'_j(K_{ab}^{\cdots a} x'^b)\right\}.$$

Contracting (4.9) with respect to indices i, k and i, j and differentiating both the equations covariantly with respect to x^k and x^j and by adding, we get

$$(4.10)\qquad \nabla_k K_{ja}^{\cdot\cdot\cdot a} + \nabla_j K_{ak}^{\cdot\cdot\cdot a} + \frac{x'^i}{n+1}\{\nabla_k R_{iaj}^{\cdot\cdot\cdot a} - \nabla_k R_{jai}^{\cdot\cdot\cdot a} + \nabla_j R_{kai}^{\cdot\cdot\cdot a} - \nabla_j R_{kai}^{\cdot\cdot\cdot a}\} + \frac{n-1}{n+1}\Big[\nabla_j K_{ka}^{\cdot\cdot\cdot a} - \nabla_k K_{ja}^{\cdot\cdot\cdot a} + \frac{1}{n+1}\{\nabla_j \dot\partial_k'(K_{ab}^{\cdot\cdot\cdot a} x'^b) - \nabla_k \dot\partial_k'(K_{ab}^{\cdot\cdot\cdot a} x'^b)\}\Big] = 0.$$

Using the relation (1.6)c in (4.10), we obtain

$$(4.11)\qquad \nabla_i K_{jk}^{\cdot\cdot\cdot i} + \frac{2x'^i}{n+1}\{\nabla_j R_{aik}^{\cdot\cdot\cdot a} - \nabla_k R_{aij}^{\cdot\cdot\cdot a}\} + \frac{n-1}{n+1}(\nabla_j K_{ka}^{\cdot\cdot\cdot a} - \nabla_k K_{ja}^{\cdot\cdot\cdot a}) = 0.$$

Differentiating (4.9) covariantly with respect to x^m and contracting the obtained relation with respect to indices i, m; we get

$$(4.12)\qquad \nabla_i W_{jk}^i = \nabla_i K_{jk}^{\cdot\cdot\cdot i} + \frac{x'^i}{n+1}\nabla_i(R_{ajk}^{\cdot\cdot\cdot a} - R_{akj}^{\cdot\cdot\cdot a}) + \frac{1}{n^2-1}(\nabla_j R_{abk}^{\cdot\cdot\cdot a} - \nabla_k R_{abj}^{\cdot\cdot\cdot a})x'^b + \frac{n-2}{n^2-1}(\nabla_j K_{ka}^{\cdot\cdot\cdot a} - \nabla_k K_{ja}^{\cdot\cdot\cdot a}).$$

Using the Bianchi identity (1.6)a for the curvature tensor $R_{jkh}^{\cdot\cdot\cdot i}(x, x')$, we get

$$(4.13)\qquad \nabla_j(R_{aml}^{\cdot\cdot\cdot a} - R_{alm}^{\cdot\cdot\cdot a})\, x'^j = (\nabla_m R_{ajl}^{\cdot\cdot\cdot a} - \nabla_l R_{ajm}^{\cdot\cdot\cdot a})\, x'^j + \nabla_a K_{lm}^{\cdot\cdot\cdot a} + (K_{jm}^{\cdot\cdot\cdot r} B_{alr}^{\cdot\cdot\cdot a} - K_{jl}^{\cdot\cdot\cdot r} B_{amr}^{\cdot\cdot\cdot a})\, x'^j.$$

With the help of equations (4.12) and (4.13), we obtain

$$(4.14) \qquad (n+1)\nabla_i W^i_{jk} = n\nabla_i K^{\cdots i}_{jk} + \frac{n}{n-1}(\nabla_j R^{\cdots a}_{aik} - \nabla_k R^{\cdots a}_{aij})x'^i$$

$$+ \frac{n-2}{n-1}(\nabla_j K^{\cdots a}_{ka} - \nabla_k K^{\cdots a}_{ja}) +$$

$$+ (K^{\cdots r}_{ij} B^{\cdots a}_{akr} - K^{\cdots r}_{ik} B^{\cdots a}_{ajr})x'^i.$$

Using equations (1.5) f, (1.7), (1.8), (4.11) and (4.14), we get

$$(4.15) \qquad (n+1)\nabla_i W^i_{jk} = -\frac{n-3}{n^2-1} n\{(\nabla_j R^{\cdots a}_{aik} - \nabla_k R^{\cdots a}_{aij})x'^i +$$

$$+ n(\nabla_j K^{\cdots a}_{ka} - \nabla_k K^{\cdots a}_{ja})\} - \frac{2}{n-1}(\nabla_j K^{\cdots k}_{ka} -$$

$$- \nabla_k K^{\cdots a}_{ja}) + (W^r_j B^{\cdots a}_{akr} - W^r_k B^{\cdots a}_{ajr}).$$

For if projective deviation tensor $W^i_j(x, x')$ vanishes, then so does W^i_{jk} and its covariant derivative, then the equation (4.15) reduces to the integrability condition (4.8) only when $\nabla_j K^{\cdots a}_{ka} - \nabla_k K^{\cdots a}_{ja} = 0$. Thus, we get

Theorem (4.2). *In a projectively flat $K^{(1)}_n$ the integrability condition* (4.8) *exists only when* $(\nabla_j K^{\cdots a}_{ka} - \nabla_k K^{\cdots a}_{ja})$ *vanishes.*

Analogous to the idea of S. KAWAGUCHI [3] the conformal curvature tensor $C^{*i}_{jkh}(x, x')$ is defined by

$$(4.16) \qquad W^i_k = C^{*i}_{jkm} x'^k x'^m.$$

Hence, we have:

Corollary (4.1). *In a conformally flat $K^{(1)}_n$* (i.e. $C^{*i}_{jkh} = 0$) *the integrability condition* (4.8) *exists only when* $(\nabla_j K^{\cdots a}_{ka} - \nabla_k K^{\cdots a}_{ja}) = 0$.

REFERENCES

[1] KAWAGUCHI, A. : *Die Geometrie des Integrals* $\int (A_i x''^i + B)^{1/p} dt$, Proc. of Imp. Acad. 12 (1936), 205-208.

[2] KAWAGUCHI, S. : *On a special Kawaguchi space of recurrent curvature*, Tensor N.S. **15**, 2 (1964), 145-158.

[3] KAWAGUCHI, S. : *On some properties of projective curvature tensor in a special Kawaguchi space*, Tensor, N.S. **13** (1963), 83-88.

[4] KANO, C. : *Conformai geometry in an* n-*dimensional space with arc length* $s = \int (A_i x''^i + B)^{1/p} dt$, Tensor N.S. **5**, 3 (1956), 187-196.

[5] OKUMURA, M. : *On some remarks of special Kawaguchi space*, Tensor N.S. **11** (1961), 154-160.

[6] PANDE, H. D. : *The conformai transformation in a Finsler space*, Mathematische Nachrichten **41** (1969), 247-252.

[7] PANDE, H.D. AND SINGH, B. : *On projective recurrent Finsler spaces of the first order*, Acad. Naz., 24-25, IV (1974), 3-8.

[8] RUND, H. : *The differential geometry of Finsler space*, SPRINGER VERLAG (1959).

ÖZET

Bu çalışmanın amacı, projektif düz özel Kawaguchi uzayının konform özellikleriyle ilgili bazı özdeşlikler bularak onları incelemektir.

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