MATRIX TRANSFORMATIONS IN SOME SEQUENCE SPACES

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Summary : The object of this note is to determine necessary and sufficient conditions for some matrix $A = (a_{nk})$ such that A-transform of $x = (x_k)$ belongs to the set $\hat{c}(q)$, where in particular $x \in c_0(p)$.

BAZI DİZİ UZAYLARINDA MATRİS DÖNÜŞÜMLERİ

Özet : Bu notun amacı, özel olarak $x \in c_0(p)$ olduğunda $x = (x_k)$ nın A-dönüşümünün $\hat{c}(q)$ cümlesine ait olması için $A = (a_{nk})$ matrisinin sağlaması gereken gerek ve yeter koşulları belirtmektir.

1. INTRODUCTION

Let s be the set of all real or complex sequences and let l_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ respectively, normed as usual by $\| x \| = \sup_k | x_k |$.

Let D be shift operator on s, that is,

$$D(x_k) = (x_{k+1}).$$

It may be recalled that the Banach limit L is a nonnegative linear functional on l_∞ such that L is invariant under the shift operator, that is, $L(Dx) = L(x)$ for all $x \in l_\infty$ and that $L(e) = 1$, where $e = (1,1,...)$. A sequence $x \in l_\infty$ is said to be almost convergent (see, Lorentz [1]) if all Banach limits of x coincide. Let $\hat{c}$ denote the set of all almost convergent sequences. Lorentz [1] proved that

$$\hat{c} = \left\{ x : \lim_{m\to\infty} t_{mn}(x) = \lim_{m\to\infty} \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \text{ exist uniformly in } n \right\}.$$

Let $p = (p_m)$ be a sequence of real numbers such that $p_m > 0$ and $\sup p_m < \infty$. We define (see, Nanda [5])

$$\widehat{c}(p) = \left\{ x : \lim_{m \to \infty} \left| \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} - L \right|^{p_m} = 0 \text{ uniformly in } n \text{ for some } L \in C \right\}.$$

If $p_m = 1 \quad \forall\, m$, then $\widehat{c}(p)$ is same as $\widehat{c}$.

Let us write $\sum x_k$ for $\sum_{k=0}^{\infty} x_k$. If $p_k > 0$ is real then $p = (p_k)$ is such that $\sup_k p_k < \infty$.

Let us list the required sequence spaces as follows [2], [3]:

$$c_0(p) = \{x = (x_k) : |x_k|^{p_k} \to 0, \quad \text{as } k \to \infty\}$$

$$c\ (p) = \{x = (x_k) : |x_k - L|^{p_k} \to 0, \quad \text{as } k \to \infty \text{ for some } L\}.$$

When $p_k = 1$ for all k we write $c_0(p) = c_0$ and $c(p) = c$.

Let X and Y be two nonempty subsets of s. Let $A = (a_{nk})$ $(n, k = 1, 2, 3, \ldots)$ be an infinite matrix of complex numbers. We write $A\,x = (A_n(x))$ if $A_n(x) = \sum_{k=0}^{\infty} a_{nk}\, x_k$ converges for each n. If $x = (x_k) \in X \Longrightarrow A\,x = (A_n(x)) \in Y$, we say that A defines a (matrix) transformation from X into Y and denote it by $A : X \to Y$. By (X, Y) we mean the class of matrices A such that $A : X \to Y$.

We now characterize the matrices in the class $(c_0(p), \widehat{c}(q))$. We write

$$t_{mn}(A\,x) = \sum_{k=0}^{\infty} a(n, k, m)\, x_k$$

where

$$a(n, k, m) = \frac{1}{m+1} \sum_{i=0}^{m} a_{n+i,\,k}$$

(this notation is used throughout).

Recently, Sridhar [9] proved the following theorem:

Theorem A. $A \in (c_0(p), \widehat{c}_0(q))$ if and only if

$$\lim_{m \to \infty} |a(n, k, m)|^{q_m} = 0 \quad (n, k = 0, 1, \ldots)$$

and

$$\lim_N \limsup_m \Big(\sum_k |a(n,k,m)| N^{-1/p_k}\Big)^{q_m} = 0 \quad (n = 0, 1, \ldots).$$

The space $\widehat{l_\infty} = \{x : \sup_{m,n} |t_{mn}(x)| < \infty\}$ has been introduced and investigated in Nanda [6]. Recently, Nanda [7] observed that this concept coincides with l_∞ as following:

$$\sup_{m,n} \left| \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \right| \geqslant \sup_n |x_n|$$

and

$$\sup_{m,n} \left| \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \right| \leq \sup_n |x_n| \; \frac{1}{m+1} \sum_{i=0}^{m} 1 = \sup_n |x_n|.$$

The object of this paper is to obtain necessary and sufficient conditions to characterize the matrices of class $A \in (c_0(p), \widehat{c}(q))$.

Now let us quote some known results as following:

Lemma B ([4]). $c_0^*(p) = \bigcup_{M>1} \Big\{(a_k) \sum_k |a_k| M^{-1/p_k} < \infty \Big\}$

Lemma C ([6], [7]). $A \in (c_0(p), l_\infty) = (c_0(p), \widehat{l_\infty})$ if and only if $\sup_m \sum_k |a(n,k,m)| M^{-1/p_k} < \infty$ for some integer $M > 1$ and n.

2. MAIN RESULT

Theorem 1. $A \in (c_0(p), \widehat{c}(q))$ if and only if

(1) $\sup_m \sum_k |a(n,k,m)| M^{-1/p_k} < \infty$ for some integer M and n,

(2) there exists $\alpha_1, \alpha_2, \ldots$ such that

$$|a(n,k,m) - \alpha_k|^{q_m} \to 0 \quad \text{as } m \to \text{uniformly in } n,$$

(3) $\lim_M \limsup_m \Big(\sum_k [a(n,k,m) - \alpha_k| M^{-1/p_k}\Big)^{q_m} = 0$ for each n.

Proof. Sufficiency: Let conditions (1) - (3) hold and $(x_k) \in c_0(p)$. Since $q = (q_m)$ is bounded, (3) and Lemma B imply in particular $(a(n,k,m) - \alpha_k) \in c_0^*(p)$ for each n and m. Now, condition (1) together with lemma B gives that $a(n,k,m) \in c_0^*(p)$ for each n and m.

Since it is clear from Lemma **B** that $c_0^*(p)$ is a linear space, on substraction, we get $(\alpha_k) \in c_0^*(p)$ so that $\sum_k \alpha_k x_k$ must exist for each $x = (x_k) \in c_0(p)$. Let $\sum_k \alpha_k x_k = L(x)$. By virtue of conditions (2), (3) and Lemma A, we have for every $x \in c_0(p)$

$$A x - L(x) \in \widehat{c}_0(q),$$

i.e.

$$A x \in \widehat{c}(q)$$

with $\lim A x = L(x)$. Hence $A \in (c_0(p), \widehat{c}(q))$.

Necessity : Let $A \in (c_0(p), \widehat{c}(q))$. Since $\widehat{c}(q) < l_\infty$ we have $A \in (c_0(p), l_\infty)$ so that by Lemma C, (1) is necessary. Condition (2) follows from the fact that $e_k \in c_0(p)$ and

$$| t_{mn}(A x) - L(x) |^{q_m} \to 0 \quad (m \to \infty, \text{ uniformly in } n). \tag{*}$$

To prove (3), let $g(x) = \sup_k | x_k |^{p_k/H}$, where $H = \max(1, \sup_k p_k)$ define a paranorm on $c_0(p)$. Then,

$$g\left(x - \sum_{k-1}^{n} x_k e_k\right) = \sup_{k \geq n+1} | x_k |^{p_k/H} \to 0 \quad (n \to \infty),$$

so that $x = \sum_k x_k e_k$ with the same topology on $c_0(p)$. Now for each n, m, $a(n, k, m)\ x_k \in c_0^*(p)$ and hence $t_{mn}(A x) \in c_0(p)$. Using uniform boundedness principle we see that $L \in c_0'(p)$. Hence for each $x \in c_0(p)$ $L(x) = L\left(\sum_k x_k e_k\right) = \sum_k x_k L(e_k)$ and by (*) we get $\left|\left(\sum_k a(n, k, m) - \alpha_k\right) x_k\right|^{q_m} \to 0$ $(m \to \infty$, uniformly in $n)$. Hence $A x - L(x) \in \widehat{c}_0(q)$, so that by Lemma A condition (3) is necessary.

Now we conclude this paper by stating the following result:

Corollary ([5]). $A \in (c_0(p), \widehat{c})$ if and only if there exists an absolute constant $M > 1$ such that

$$\sup_m \sum_k |a(n, k, m)| M^{-1/p_k} < \infty \quad \text{for each } n$$

and

$$\lim_{m \to \infty} a(n, k, m) = \alpha_k \text{ uniformly in } n, \text{ for each fixed } k.$$

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