AN F-PERIODIC FUNCTION AND A PLANE CURVE

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Summary : In this paper F-periodic functions are defined. The Fourier series for F-periodic functions are considered. Applications of F-periodic functions to plane curves are given.

BİR F-PERİYODİK FONKSİYON VE BİR DÜZLEM EĞRİSİ

Özet : Bu çalışmada F-periyodik fonksiyonlar tanımlanmakta ve bunların Fourier serileri gözönüne alınmakta, ayrıca bu tür fonksiyonların düzlem eğrilerine uygulamaları verilmektedir.

INTRODUCTION

A real function f defined on $\mathbf{R} = (-\infty, +\infty)$ is called F-periodic with respect to a certain special function F if the equality holds:

$$f(F(t)) = f(t).$$

Next we define a Fourier series for these functions. In the second part of the paper we apply F-periodic function to a plane curve.

Namely, let a plane curve be represented by (compare (16) [1])

$$r_{f,k}(s) = \int_{s_0}^{s} f(t)\, k(t)\, e^{iK(t)}\, dt, \quad s \in \mathbf{R}$$

and let the Fourier series (15) [1] for f be given by

$$f(s) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{2\pi}{\beta} nK(s) + B_n \sin \frac{2\pi}{\beta} nK(s) \right).$$

Then the coefficients of the series are invariants of the curve.

1. ON AN F-PERIODIC FUNCTION

Let (a, b) denote an opened interval included in the real line $\mathbf{R} = (-\infty, +\infty)$. We consider a one-to-one map $F\colon (a, b) \longleftrightarrow (a, b)$ such that

(A) $t < F(t)$ for all $t \in (a, b)$,

(B) F is a strictly increasing function,

(C) F has the derivative on the whole interval (a, b).

Definition 1. Let us fix the function F. A function $f\colon (a, b) \longrightarrow \mathbf{R}$ will be called *F-periodic* if and only if

$$f(F(t)) = f(t) \text{ for all } t \in (a, b). \tag{1.1}$$

If $F(t) = t + L$, $0 \neq L \in \mathbf{R}$, then the above-mentioned definition gives usual periodic function.

Now we prove that there exists an F-periodic function.

Theorem 1. If the function F is given, then there exists an F-periodic function different from a constant.

Proof. To prove it, we define

$$F^{\nu}(t) = \underbrace{F(F(\ldots(F}_{\nu-\text{times}}(t))\ldots))$$

and

$$F^{-\nu}(t) = \underbrace{F^{-1}(F^{-1}(\ldots(F^{-1}}_{\nu-\text{times}}(t))\ldots))$$

for $\nu = 1, 2, 3, \ldots$, where F^{-1} is the inverse function for F and $F^{0}(t) = t$.

By (A) and (B) for each fixed number $t_0 \in (a, b)$ the interval is the following sum

$$(a, b) = \bigcup_{\nu=-\infty}^{\nu=+\infty} \langle F^{\nu}(t_0), F^{\nu+1}(t_0)), \tag{2.1}$$

where $\langle r, s) = \{\lambda \in \mathbf{R} : r \leq \lambda < s\}$.

Let $\gamma\colon \langle t_0, F(t_0)) \longrightarrow \mathbf{R}$ be a real function. We observe that for an arbitrary $t \in (a, b)$ there exists exactly one number $s \in \langle t_0, F(t_0))$ such that $t = F^{\nu}(s)$ for exactly one integer ν. Hence we define

$$\alpha(t) \overset{\text{def}}{=} \gamma(s) \tag{3.1}$$

for all $t \in (a, b)$. Obviously $\alpha(F(t)) = \alpha(t)$, because $F(t) = F^{\nu+1}(s)$. This means that α is an F-periodic function.

F-periodic functions have properties similar to usual periodic functions. Clearly, if f, g are two F-periodic functions, then $f \pm g, fg$ and $\frac{f}{g}$ (whenever $g \neq 0$) are F-periodic functions, too. For different functions F we can consider different F-periodic functions.

Example. The map $t \longrightarrow t^2$ with the domain $(0,1)$ or $(1, +\infty)$ satisfies the assumptions (A) and (B). Thus there exists a t^2-periodic function f different from a constant, i.e. $f(t^2) = f(t)$.

Theorem 2. If F is a continuous function, then there exists a continuous F-periodic function different from a constant.

Proof. We consider decomposition (2.1) of the interval (a, b). Next taking the continuous function $\gamma\colon < t_0 , F(t_0)) \longrightarrow \mathbf{R}$ such tahat

$$\lim_{\substack{t \to F(t_0) \\ t < F(t_0)}} \gamma(t) = \gamma(t_0) \tag{4.1}$$

we define by (3.1) the continuous F-periodic function $t \longrightarrow \alpha(t)$.

To introduce the Fourier series for an F-periodic function we prove the following

Proposition 1. Let f be a continuous F-periodic function with respect to the function F satisfying (A) - (C) and let the function $H(s)$ be defined as follows:

$$s \longrightarrow H(s) = \int_s^{F(s)} \alpha(t) f(t)\, dt, \tag{5.1}$$

where $t \longrightarrow \alpha(t)$ is a certain function. Then the function H is constant if and only if

$$\alpha(F(t))\, F'(t) = \alpha(t) \text{ for all } t \in (a, b).$$

Proof. We have

$$H'(s) = (\alpha(F(s))\, F'(s) - \alpha(s)) f(s).$$

Proposition 2. If the function F is given and F satisfies conditions (A) - (C), then there exists a positive function α such that

$$F'(t) = \frac{\alpha(t)}{\alpha(F(t))}. \tag{6.1}$$

Proof. We apply notions from the proof of Prop. 1. Considering the decomposition (2.1) we define

$$\alpha(t) = \begin{cases} \gamma(t), \text{ if } t \in \langle t_0, F(t_0)) \\ \dfrac{\gamma(s)}{\left(\dfrac{d}{ds} F^{\nu}\right)(s)}, \text{ if } t = F^{\nu}(s), \ \nu = 1, 2, 3, \ldots \\ \gamma(s)\left(\dfrac{d}{ds} F^{\mu}\right)(s), \text{ if } t = F^{\mu}(s), \ \mu = -1, -2, -3, \ldots \end{cases} \tag{7.1}$$

where $\gamma(t)$ is a positive function.

$$\alpha(F(t)) = \begin{cases} \dfrac{\gamma(t)}{\left(\dfrac{d}{dt} F\right)(t)}, \text{ if } t \in \langle t_0, F(t_0)) \\ \dfrac{\gamma(s)}{\left(\dfrac{d}{ds} F^{\nu+1}\right)(s)}, \text{ if } t = F^{\nu}(s), \ \nu = 1, 2, 3, \ldots \\ \gamma(s)\left(\dfrac{d}{ds} F^{\mu+2}\right)(s), \text{ if } t = F^{\mu}(s), \ \mu = -2, -3, -4, \ldots \\ \gamma(s), \text{ if } t = F^{-1}(s). \end{cases}$$

Hence

$$\frac{\alpha(t)}{\alpha(F(t))} = \begin{cases} F'(t), \text{ if } t \in \langle t_0, F(t_0)) \\ F'(F^{\nu}(s)) = F'(t), \text{ if } t = F^{\nu}(s), \ \nu = 1, 2, 3, \ldots \\ F'(F^{\mu}(s)) = F'(t), \text{ if } t = F^{\mu}(s), \ \mu = -1, -2, -3, \ldots. \end{cases}$$

So it is verified that

$$F'(t) = \frac{\alpha(t)}{\alpha(F(t))}.$$

Corollary 1. If the function γ satisfies the condition (4.1), then the function $\alpha(t)$ defined by (7.1) is continuous.

Now we present the Fourier series for an F-periodic function whenever F satisfies the conditions (A) - (C). We fix a positive continuous F-periodic function $t \longrightarrow k(t)$ and we fix a positive continuous function $\alpha(t)$ satisfying the condition (6.1). By Proposition 1 the number β defined as follows

$$\beta = \int_{t}^{F(t)} \alpha(s)\, k(s)\, ds$$

is not dependent on the variable $t \in (a, b)$.

Proposition 3. If g is a function defined on **R** with the period β, then the following function

$$f(t) = g(K(t)),$$

where $K(t) = \int_{t_0}^{t} \alpha(s)\, k(s)\, ds$, $t, t_0 \in (a, b)$, is F-periodic.

Indeed

$$f(F(t)) = g(K(F(t))) = g\Big(\int_{t_0}^{F(t)} \alpha(s)\, k(s)\, ds\Big) = g\Big(\int_{t_0}^{t} + \int_{t}^{F(t)}\Big) =$$
$$= g(K(t) + \beta) = g(K(t)) = f(t).$$

For each fixed $t \in (a, b)$ we consider the real Hilbert space $L^2_{[t, F(t)]}[k\,\alpha]$ of all functions defined in the interval $[t, F(t)]$ with the scalar product:

$$(f, g) = \int_{t}^{F(t)} k(s)\, \alpha(s) f(s)\, g(s)\, ds.$$

Proposition 4. The sequence of F-periodic functions:

$$\frac{1}{\sqrt{\beta}},\ \sqrt{\frac{2}{\beta}} \cos \frac{2\pi n}{\beta} K(s),\ \sqrt{\frac{2}{\beta}} \sin \frac{2\pi n}{\beta} K(s),\ n = 1, 2, 3, \ldots \tag{8.1}$$

with the domain $s \in [t, F(t)]$ is the orthonormal and complete system in the real Hilbert space $L^2_{[t, F(t)]}[k\,\alpha]$ for each $t \in (a, b)$.

Let $f\colon (a, b) \longrightarrow$ **R** be an F-periodic continuous function. The Fourier series for f restricted to the interval $t \le s \le F(s)$ is given by the formula

$$f(s) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \Big(A_n(t) \cos \frac{2\pi}{\beta} n\, K(s) + B_n(t) \sin \frac{2\pi}{\beta} n\, K(s)\Big). \tag{9.1}$$

Proposition 5. The coefficients $A_n(t)$, $B_n(t)$ of series (5.2) are independent of the variable t.

Proof. The coefficients are expressed by the following formulas:

$$A_n(t) = \frac{2}{\beta} \int_{t}^{F(t)} f(u)\, k(u)\, \alpha(u) \cos \frac{2\pi}{\beta} n\, K(u)\, du$$

and

$$B_n(t) = \frac{2}{\beta} \int_{t}^{F(t)} f(u)\, k(u)\, \alpha(u) \sin \frac{2\pi}{\beta} n\, K(u)\, du.$$

By the relation (6.1) those coefficients are constant functions for $n = 1, 2, \ldots$.

2. APPLICATIONS TO A PLANE CURVE

Let F be a function satisfying (A) - (C). We denote by L_F the set of all positive continuous functions such that

$$F'(t) = \frac{\alpha(t)}{\alpha(F(t))} .$$

Next we denote by C_F the set of all F-periodic continuous functions defined on (a,b). In this section we examine the following class $\mathfrak{M}$ of all C^1 -curves of the form

$$z(t) = r_{\alpha,k}(t) = \int_{t_0}^{t} \alpha(s)\, e^{iK(s)}\, ds, \tag{1.2}$$

where $\alpha \in L_F$, $k \in C_F$.

Let C_L denote the set of all positive continuous periodic functions defined on $\mathbf{R}$ with the period L. If f, $p \in C_L$ and $\alpha = p f$, $k = \frac{1}{f}$, then formula (1.2) gives the representation for plane curve considered in [1].

For every curve $r_{\alpha,k} \in \mathfrak{M}$ we can compute the curvature in the following meaning: Let ω denote the angle between tangent vectors $z'(s)$ and $z'(s+h)$ and l denote the length of the arc of the curve between the points $z(s)$ and $z(s+h)$. Then there exists the bound

$$\lim_{h\to 0} \frac{\omega}{l} = k(s). \tag{2.2}$$

Indeed

$$\lim_{h\to 0} \frac{\omega}{l} = \lim_{h\to 0} \frac{\omega}{\sin\omega} \frac{\sin\omega}{l} = \lim_{h\to 0} \frac{\sin\omega}{l} =$$

$$= \lim_{h\to 0} \frac{[z'(s), z'(s+h)]}{|z'(s)|\,|z'(s+h)|\,\left|\int_s^{s+h} |z'(t)|\, dt\right|},$$

where $[z, w]$ is the determinant of the vectors z and w.

Hence

$$\lim_{h\to 0} \frac{\omega}{l} = \frac{1}{|z'(s)|^3} \lim_{h\to 0} \frac{[z'(s),\ z'(s+h)]}{h} .$$

But $z'(t) = \alpha(t)\, e^{iK(t)}$ (def $K(t)$ see Prop. 3).

Thus

$$[z'(s),\ z'(s+h)] = \alpha(s)\,\alpha(s+h)\int_s^{s+h}\alpha(u)\,k(u)\,du.$$

Finally

$$\lim_{h\to 0}\frac{\omega}{l} = \frac{1}{\alpha^3(s)}\,\alpha(s)\,\alpha(s+h)\,\frac{1}{h}\int_s^{s+h}\alpha(u)\,k(u)\,du = k(s).$$

So a counterpart of theorem about integration of the curvature has the form:

Proposition 6. Let a curve $r_{\alpha,k}\in\mathfrak{M}$ be represented by (2.3). Then the angle between tangent vectors $z'(t_1)$ and $z'(t_2)$ is equal to:

$$\int_{t_1}^{t_2} k(s)\,\alpha(s)\,ds.$$

Now we present a certain geometrical property common to all curves from the class $\mathfrak{M}$. To express it, we consider a positively oriented oval T [2]. Let X denote a point lying on Γ which moves on Γ conformable to the orientation of Γ. It is easy to see that if the point X passes the way equal to the length of Γ, then the tangent vector at the point X rotates on the angle equal to 2π. Now we show the following generalization:

Proposition 7. An arbitrary curve $r_{\alpha,k}\in\mathfrak{M}$ has the following properties:

(I) $r_{\alpha,k}$ is a locally strictly convex curve,

(II) the curve determines two numbers $m\neq 0$ and $\beta\neq 0$ such that if a point $X\in r_{\alpha,k}$ passes the way equal to m, then the tangent vector at the point X rotates on the angle equal to β.

Proof. (I) Let a plane curve $r_{\alpha,k}\in\mathfrak{M}$. Now, if $r_{\alpha,k}(t')$ is a fixed point on $r_{\alpha,k}$ then the unit tangent and normal vectors are equal to $e^{iK(t')}$ and $ie^{iK(t')}$, respectively. The vector v hooked to $r_{\alpha,k}$(t') with the end in the arbitrary point $r_{\alpha,k}(t'')$ is expressed by

$$v = \int_{t'}^{t''}\alpha(s)\,e^{iK(t)}\,ds,$$

whenever $t' < t''$.

The determinant of v with the unit vector has the following form :

$$[e^{iK(t')},\ v] = \int_{t'}^{t''}\alpha(s)\sin\left(K(s)-K(t')\right)ds.$$

Hence putting $u = K(s)$, $du = \alpha(s)\,k(s)\,ds$ we have

$$[e^{iK(t')}, v] = \int_{K(t')}^{K(t'')} \frac{1}{k(K^{-1}(u))} \sin (u - K(t'))\, du,$$

where K^{-1} denotes the inverse function to K.

Next putting $\lambda = u - K(t')$, $d\lambda = du$, we have

$$[e^{iK(t')}, v] = \int_{0}^{K(t'')-K(t')} \frac{1}{k(K^{-1}(\lambda + K(t')))} \sin \lambda \, d\lambda.$$

This means that the determinant is non-negative. Thus $r_{\alpha,k}(t')$ and $r_{\alpha,k}(t'')$ are in the same half-plane determined by the tangent vector $e^{iK(t')}$. Therefore $r_{\alpha,k}$ is a locally convex curve.

(II) By Prop. 6 if a point X moves on $r_{\alpha,k}$ and passes the way equal to

$$m = \int_{t}^{F(t)} \alpha(s)\, ds,$$

then the tangent vector at X rotates on the angle equal to $\beta = \int_{t}^{F(t)} \alpha(s)\, k(s)\, ds$.

Let a plane curve $r_{\alpha,k} \in \mathfrak{M}$ be represented by formula (2.3). Obviously $\frac{1}{k}$ is F-periodic function and it has the Fourier series in the form

$$\frac{1}{k} = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{2\pi}{\beta} n K + B_n \sin \frac{2\pi}{\beta} n K \right). \tag{3.2}$$

It is easy to verify that if $r_{\alpha,k} \in \mathfrak{M}$ is a closed curve, then the perimeter of $r_{\alpha,k}$ is equal to πA_0. Thus the coefficient A_0 is an invariant of the curve. Generalizing this fact we give the following

Theorem 3. For each curve $r_{\alpha,k}$ the Fouries coefficients of series (1.6) are invariants for the curve.

Proof. This means that A_n, B_n $(n=0,1,2,\ldots)$ do not depend on translation and rotation and parameterization. The proof of the first part of the theorem is easy and we omit it. To prove that the coefficients do not depend on parameterization we consider two equivalent curves. Let a one-to-one function $\sigma: (c, d) \longleftrightarrow (a,b)$ be given. We assume that $\sigma'(\tau) > 0$ for $\tau \in (c. d)$. Then the curve $r_{\alpha,k}(\sigma(\tau))$ is equivalent to the curve $r_{\alpha,k}(t)$. Exactly $r_{\alpha,k}(\sigma(\tau)) \in \mathfrak{M}$, i.e.:

$$r_{\alpha,k}(\sigma(\tau)) = r_{\alpha_1,k_1}(\tau),$$

where $\alpha_1(\tau) = \alpha(\sigma(\tau))\,\sigma'(\tau)$ and $k_1(\tau) = k(\sigma(\tau))$. Indeed, with the aid of the standard change of variable rule $s = \sigma(\mu)$ we obtain

$$r_{\alpha,k}(\sigma(\tau)) = \int_{t_0}^{\sigma(\tau)} \alpha(s)\, e^{i\int_{t_0}^{s} \alpha(u)\,k(u)\,du}\, ds = \int_{\tau_0}^{\tau} \alpha(\sigma(\mu))\,\sigma'(\mu)\, e^{i\int_{t_0}^{\sigma(\mu)} \alpha(u)\,k(u)\,du}\, d\mu\,;$$

secondly we put $u = \sigma(\rho)$ and $du = \sigma'(\rho)\,d\rho$

$$= \int_{\tau_0}^{\tau} \alpha(\sigma(\mu))\,\sigma'(\mu)\, e^{i\int_{t_0}^{\mu} \alpha(\rho)\,k(\rho)\,\sigma'(\rho)\,d\rho}\, d\mu.$$

Denoting $\alpha_1(\tau) = \alpha(\sigma(\tau))\,\sigma'(\tau)$ and $k_1(\tau) = k(\sigma(\tau))$ we have

$$r_{\alpha,k}(\sigma(\tau)) = r_{\alpha_1,k_1}(\tau).$$

It is easy to verify that $k_1 \in C_{F_1}$, $\alpha_1 \in L_{F_1}$, where $F_1 = \sigma^{-1}(F(\sigma(\tau)))$.
Let a curve $\Lambda \in \mathfrak{M}$ have two equivalent representations: $z = r_{\alpha,k}(t)$ and $z = r_{\alpha_1,k_1}(\tau)$, where $t = \sigma(\tau)$. Let $A_0^{(1)}$, $A_n^{(1)}$, $B_n^{(1)}$ denote the Fourier coefficients for $\frac{1}{k_1}$. Then we obtain the equalities

$$A_0 = A_0^{(1)},\ A_n = A_n^{(1)},\ B_n = B_n^{(1)},\ n = 1, 2, \ldots .$$

By the definition of the Fourier coefficient we have

$$A_n^{(1)} = \frac{2}{\beta} \int_{\tau}^{F_1(\tau)} \alpha_1(\nu) \cos \frac{2\pi}{\beta}\, n\, K_1(\nu)\, d\nu =$$

$$= \frac{2}{\beta} \int_{\tau}^{\sigma^{-1}(F(\sigma(\tau)))} \alpha(\sigma(\nu)) \cos\left(\frac{2\pi}{\beta}\, n \int_{\tau_0}^{\nu} \alpha(\sigma(\rho))\, k(\sigma(\rho))\, \sigma'(\rho)\, d\rho\right) \sigma'(\nu)\, d\nu,$$

now we change variable putting $r = \sigma(\rho)$ and $dr = \sigma'(\rho)\,d\rho$

$$= \frac{2}{\beta} \int_{\tau}^{\sigma^{-1}(F(\sigma(\tau)))} \alpha(\sigma(\nu)) \cos\left(\frac{2\pi}{\beta}\, n \int_{\sigma(\tau_0)}^{\sigma(\nu)} \alpha(r)\, k(r)\, dr\right) \sigma'(\nu)\, d\nu,$$

next we put $\mu = \sigma(\nu)$ and $d\mu = \sigma'(\nu)\,d\nu$

$$= \frac{2\pi}{\beta} \int_{\sigma(\tau)}^{F(\sigma(\tau))} \alpha(\mu) \cos\left(\frac{2\pi}{\beta} n \int_{t_0}^{\mu} \alpha(r)\, k(r)\, dr\right) d\mu\, ;$$

but $K(\mu) = \int_{t_0}^{\mu} \alpha(r)\, k(r)\, dr$, so finally we obtain

$$= \frac{2}{\beta} \int_{t}^{F(t)} \alpha(\mu) \cos \frac{2\pi}{\beta} n\, K(\mu)\, d\mu = A_n\, .$$

Similarly we verify the equalities $A_0^{(1)} = A_0$ and $B_n^{(1)} = B_n$. This means that the Fourier coefficients of $\frac{1}{k}$ are independent of parameterizations. This completes the proof.

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