İstanbul Üniv. Fen Fak. Mat. Der. 52 (1993), 17-21

ON GENERALIZED CO-SYMPLECTIC MANIFOLD

S.N. PANDEY - R.H. OJHA
Department of Mathematics, Faculty of Science, Hindu University, Varanasi 221005, INDIA

Summary : In the previous paper [1], the authors studied a semi-symmetric connexion on generalized cosmyplectic manifold. The purpose of the present paper is the investigation of some properties of this manifold. The curvature identities are also obtained.

GENELLEŞTİRİLMİŞ "CO-SYMPLECTIC" MANİFOLD

Özet : Bu çalışmada, yazarlar tarafından daha önce incelenmiş olan, genelleştirilmiş "co-symplectic" manifoldun bazı özelikleri araştırılmakta ve eğrilik özdeşlikleri elde edilmektedir.

1. INTRODUCTION

In an n-dimensional differentiable manifold M_n of class C^∞, a vector valued real linear function F, a 1-form u and U is a vector field satisfying

$$\text{a)}\quad \bar{\bar{X}} = -X + u(X)U, \qquad \text{b)}\quad \bar{X} = F(X). \tag{1.1}$$

Then M_n is an almost contact manifold and $\{F, U, u\}$ is said to give an almost contact structure to M_n. In an almost contact manifold $\bar{U} = 0$, $u(\bar{X}) = 0$, $u(U) = 1$, $n = 2m + 1$.

Let the almost contact manifold M_n be endowed with the metric tensor g satisfying

$$g(\bar{X}, \bar{Y}) = g(X, Y) - u(X)\,u(Y). \tag{1.2}$$

Then M_n is called an almost contact metric manifold. An almost contact metric manifold satisfying

$$(D_X\,'F)\;(Y, Z) = u(Y)\;(D_X\,u)\;(\bar{Z}) - u(Z)\;(D_X\,u)\;(\bar{Y}) \tag{1.3}$$

is called generalized cosymplectic manifold [2].

The Nijenhuis tensor in generalized cosymplectic manifold is given by

$$
\begin{aligned}
&\text{a)}\quad N(X, Y) = (D_{\bar{X}} F)(Y) - (D_{\bar{Y}} F)(X) - \overline{(D_X F)(\bar{Y})} - \overline{(D_Y F)(\bar{X})},\\
&\text{b)}\quad {}'N(X, Y, Z) = (D_{\bar{X}}\, {}'F)(Y, Z) - (D_{\bar{Y}}\, {}'F)(X, Z) + \\
&\qquad + (D_X\, {}'F)(Y, \bar{Z}) - (D_Y\, {}'F)(X, \bar{Z}).
\end{aligned} \tag{1.4}
$$

2. GENERALIZED COSYMPLECTIC MANIFOLD

Theorem 2.1. Let D be a Riemannian connexion on an almost contact metric manifold and let B be an affine connexion satisfying $D_X\, {}'F = 0$. Then M_n is a generalized cosymplectic manifold iff

$$
{}'H(X, \bar{\bar{Y}}, \bar{\bar{Z}}) + {}'H(X, \bar{Z}, \bar{Y}) = 0 \tag{2.1}
$$

where ${}'H(X, Y, Z) = g(H(X, Y), Z)$.

Proof. $(D_X\, {}'F) = 0 \Longrightarrow (D_X\, {}'F)(Y, Z) = {}'H(X, Z, \bar{Y}) - {}'H(X, Y, \bar{Z})$
$\Longrightarrow (D_X\, {}'F)(\bar{Y}, \bar{Z}) = {}'H(X, \bar{Z}, \bar{\bar{Y}}) - {}'H(X, \bar{Y}, \bar{\bar{Z}})$.

Using (2.1) we get $(D_X\, {}'F)(\bar{Y}, \bar{Z}) = 0$.

$$
(D_X\, {}'F)(\bar{Y}, \bar{Z}) = 0
$$

is equivalent to

$$
(D_X\, {}'F)(\bar{Y}, \bar{\bar{Z}}) = 0
$$

or

$$
(D_X\, {}'F)(-Y + u(Y)U, -Z + u(Z)U) = 0
$$

or

$$
(D_X\, {}'F)(Y, Z) - u(Y)(D_X\, {}'F)(U, Z) - u(Z)(D_X\, {}'F)(Y, U) = 0
$$

or

$$
(D_X\, {}'F)(Y, Z) = u(Y)(D_X u)(\bar{Z}) - u(Z)(D_X u)(\bar{Y}),
$$

which is the definition of generalized cosymplectic manifold.

Theorem 2.2. F is Killing iff

$$
(D_X u)(\bar{Z}) = 0.
$$

Proof. Since F is Killing we have

$$
(D_X\, {}'F)(Y, Z) + (D_Y\, {}'F)(X, Z) = 0
$$

or

$$
u(Y)(D_X\, {}'F)(U, Z) + u(X)(D_Y\, {}'F)(U, Z) + u(Z)\{(D_X\, {}'F)(Y, U) + (D_Y\, {}'F)(X, U)\} = 0
$$

or

$$u(Y)\ (D_X{}'F)\ (U, Z) + u(X)\ (D_Y{}'F)\ (U, Z) = 0$$

or

$$u(Y)\ (D_X u)\ (\bar{Z}) + u(X)\ (D_Y u)\ (\bar{Z}) = 0.$$

Putting U for Y we get

$$(D_X u)\ (\bar{Z}) + u(X)\ (D_U u)\ (\bar{Z}) = 0. \tag{2.2}$$

Putting U for X in this equation we obtain

$$(D_U u)\ (\bar{Z}) = 0. \tag{2.3}$$

From (2.2) and (2.3) we get the result.

The converse is obvious.

Theorem 2.3. If U is Killing then

$$'N(X, Y, Z) - d'F(X, Y, \bar{Z}) = 2u(Z)\ (D_{\bar{Y}} u)\ (\bar{X}). \tag{2.4}$$

Proof. From (1.4) b) we have

$$'N(X, Y, Z) - d'F(X, Y, \bar{Z}) = (D_{\bar{X}}{}'F)\ (Y, Z) - (D_{\bar{Y}}{}'F)\ (X, Z) - $$
$$- (D_{\bar{Z}}{}'F)\ (X, Y).$$

Using (1.3) in this equation we get

$$'N(X, Y, Z) - d'F(X, Y, \bar{Z}) = - u(X)\ ((D_{\bar{Y}} u)\ (\bar{Z}) + (D_{\bar{Z}} u)\ (\bar{Y})) +$$
$$+ u(Y)\ ((D_{\bar{X}} u)\ (\bar{Z}) + (D_{\bar{Z}} u)\ (\bar{X})) +$$
$$+ u(Z)\ ((D_{\bar{Y}} u)\ (\bar{X}) - (D_{\bar{X}} u)\ (\bar{Y})).$$

Since U is Killing then

$$'N(X, Y, Z) - d'F(X, Y, \bar{Z}) = 2u(Z)\ (D_{\bar{Y}} u)\ (\bar{X}).$$

Corollary 2.1. If $'F$ is closed then

$$'N(X, Y, \bar{Z}) = 0. \tag{2.5}$$

Proof. Putting $d'F = 0$ in (2.4) we get at once (2.5).

Theorem 2.4. A generalized cosymplectic manifold is quasi-Sasakian if

$$(D_X{}'F)\ (Y, U) = (D_Y{}'F)\ (X, U). \tag{2.6}$$

Proof. By (1.3) we have

$$(D_X{}'F)(Y, Z) + (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y) =$$
$$= u(Y)(D_X u)(\bar{Z}) - u(Z)(D_X u)(\bar{Y}) + u(Z)(D_Y u)(\bar{X}) -$$
$$- u(X)(D_Y u)(\bar{Z}) + u(X)(D_Z u)(\bar{Y}) - u(Y)(D_Z u)(\bar{X}) =$$
$$= u(X)((D_Z u)(\bar{Y}) - (D_Y u)(\bar{Z})) + u(Y)((D_X u)(\bar{Z}) - (D_Z u)(\bar{X})) +$$
$$+ u(Z)((D_Y u)(\bar{X}) - (D_X u)(\bar{Y})).$$

Using (2.6) in the above equation we get

$$(D_X{}'F)(Y, Z) + (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y) = 0,$$

which proves the statement.

3. CURVATURE IDENTITIES

We have

$$(D_Y{}'F)(Z, W) = u(Z)(D_Y u)(\bar{W}) - u(W)(D_Y u)(\bar{Z}).$$

Consequently

$$(D_X D_Y{}'F)(Z, W) + (D_Y{}'F)(D_X Z, W) + (D_Y{}'F)(Z, D_X W) =$$
$$= ((D_X u)(Z) + u(D_X Z))(D_Y u)(\bar{W}) + u(Z)((D_X D_Y u)(\bar{W}) +$$
$$+ (D_Y u)(D_X \bar{W})) -$$
$$- ((D_X u)(W) + u(D_X W))(D_Y u)(\bar{Z}) - u(W)((D_X D_Y u)(\bar{Z}) +$$
$$+ (D_Y u)(D_X \bar{Z})$$

or

(i) $$(D_X D_Y{}'F)(Z, W) = u(Z)(D_X D_Y u)(\bar{W}) - u(W)(D_X D_Y u)(\bar{Z}) +$$
$$+ (D_X u)(Z)(D_Y u)(\bar{W}) - (D_X u)(W)(D_Y u)(\bar{Z}) +$$
$$+ u(Z)(D_X F)(W) - u(W)(D_X F)(Z)$$

(ii) $$-(D_Y D_X{}'F)(Z, W) = -u(Z)(D_Y D_X u)(\bar{W}) + u(W)(D_Y D_X u)(\bar{Z}) -$$
$$- (D_Y u)(Z)(D_X u)(\bar{W}) + (D_Y u)(W)(D_X u)(\bar{Z}) -$$
$$- u(Z)(D_Y F)(W) + u(W)(D_Y F)(Z)$$

(iii) $$-(D_{[X,Y]}{}'F)(Z, W) = -u(Z)(D_{[X,Y]} u)(\bar{W}) + u(W)(D_{[X,Y]} u)(\bar{Z}).$$

Adding the above three we get

$$(D_X D_Y{}'F)(Z, W) - (D_Y D_X{}'F)(Z, W) - (D_{[X,Y]}{}'F)(Z, W) =$$
$$= u(Z)((D_X D_Y u)(\bar{W}) - (D_Y D_X u)(\bar{W}) - (D_{[X,Y]} u)(\bar{W})) -$$
$$- u(W)((D_X D_Y u)(\bar{Z}) - (D_Y D_X u)(\bar{Z}) - (D_{[X,Y]} u)(\bar{Z})) +$$

$$+ (D_X u)(Z)(D_Y u)(\overline{W}) - (D_X u)(W)(D_Y u)(\bar{Z}) +$$
$$+ u(Z)(D_X F)(W) - u(W)(D_X F)(Z) -$$
$$- (D_Y u)(Z)(D_X u)(\overline{W}) + (D_Y u)(W)(D_X u)(\bar{Z}) -$$
$$- u(Z)(D_Y F)(W) + u(W)(D_Y F)(Z)$$

or

$$'K(X, Y, Z, \overline{W}) + 'K(X, Y, \bar{Z}, W) = - u(Z)('K(X, Y, \overline{W}, U)) +$$
$$+ u(W)('K(X, Y, \bar{Z}, U)) + A(X, Z, Y, W) - A(Y, Z, X, W)$$

where

$$A(X, Z, Y, W) = (D_X u)(Z)(D_Y u)(\overline{W}) - (D_X u)(W)(D_Y u)(\bar{Z}) +$$
$$+ u(Z)(D_X F)(W) - u(W)(D_X F)(Z).$$

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