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CONTACT RIEMANNIAN MANIFOLDS SATISFYING $R(\xi, X) . \bar{C} = 0$

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Summary : The object of this paper is to characterize a contact metric manifold satisfying $R(\xi, X) . \bar{C} = 0$ where $\bar{C}$ is the conharmonic curvature tensor and $R(\xi, X)$ denotes the derivation of the tensor algebra at each point of the tangent space.

$R(\xi, X) . \bar{C} = 0$ BAĞINTISINI GERÇEKLEYEN KONTAKT KIEMANN MANİFOLDLARI

Özet : Bü çalışmada, $\bar{C}$ konharmonik eğrilik tensörünü ve $R(\xi, X)$ tensör cebrinin teğet uzayın her bir noktasındaki türevini göstermek üzere, $R(\xi, X) . \bar{C} = 0$ bağıntısını gerçekleyen bir kontakt metrik manifold karakterize edilmektedir.

1. INTRODUCTION

In this paper we consider a contact metric manifold M^{2m+1} (ϕ, η, ξ, g) with characteristic vector field ξ belonging to the **K**-nullity distribution. If ξ is a killing vector field then the M^{2m+1} is said to be Sasakian. In a recent paper [3] the first author and N. Guha proved that a Sasakian manifold satisfying $R(X, Y) . \bar{C} = 0$ where $R(X, Y)$ denotes the derivation of the tensor algebra at each point of the tangent space and $\bar{C}$ is the conharmonic curvature tensor [4] is locally isometric to the unit sphere S^{2m+1} (1). In section 2 of this paper we extend this result to contact metric manifolds and prove that either M^{2m+1} is locally isometric to the Riemannian product E^{m+1} X S^m (4) or M^{2m+1} is locally isometric to S^{2m+1} (1). Contact Riemannian manifold satisfying $R(X, \xi) . R = 0$ has been studied by Perrone [6].

2. A contact manifold is a C^∞ $(2m + 1)$ manifold M^{2m+1} equipped with a global 1-form η such that $\eta X (d\eta)^m \neq 0$ everywhere on M^{2m+1}. Given a contact form η it is well-known that there exists a unique vector field ξ on M^{2m+1} satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X on M^{2m+1}. A Riemannian metric g is said to be an associated metric if there exists a tensor field ϕ of type (1.1) such that

$$d\eta(X, Y) = g(X, \phi Y), \ \eta(X) = g(X, \xi)$$

and

$$\phi^2 = -I + \eta(X)\xi.$$

The structure (ϕ, η, ξ, g) on M^{2m+1} is called a contact metric structure and M^{2m+1} equipped with such a structure is said to be a contact metric manifold. We refer the reader to [1] as a general reference for the ideas of this paragraph. Denoting by L Lie differentiation, we define a tensor field h by $h = \frac{1}{2} L_\xi \phi$. h is symmetric and satisfies $\phi h = -h\phi$. So if λ is an eigen value of h with eigen vector X, $-\lambda$ is also an eigen value with eigen vector ϕX. We also have $T_r h = T_r \phi h = 0$ and $h\xi = 0$. Moreover if ∇ denotes the Riemannian connection of g, the following formulas hold

$$\nabla_X \xi = -\phi X - \phi h X \tag{2.1}$$

$$\nabla_\xi \phi = 0 \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

The vector field ξ is killing with respect to g if and only if $h = 0$. A contact metric manifold $M^{2m+1}(\phi, \eta, \xi, g)$ for which ξ is killing is said to be a k-contact manifold. If the almost complex structure J on $M^{2m+1} X R$, defined by $J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$ where f is a real-valued function, is integrable then the structure is said to be normal and $M^{2m+1}(\phi, \eta, \xi, g)$ is said to be Sasakian. If R denotes the curvature tensor, a Sasakian manifold may be characterized by $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$. A Sasakian manifold is k-contact, but the converse is true only when dim $M^{2m+1} = 3$.

The k-nullity distribution [7] of a Riemannian manifold (M, g) for a real number K is a distribution

$$N(k) : p \to N_p(k) = \{Z \in T_p M / R(X, Y)\, Z = k\,[g(Z, Y)\, X - g(X, Z)\, Y]\}$$

for any $X, Y \in T_p(M)$. Suppose that $M^{2m+1}(\phi, \eta, \xi, g)$ is a contact metric manifold with ξ belonging to the K nullity distribution. That is,

$$R(X, Y)\,\xi = K[\eta(Y)\, X - \eta(X)\, Y]. \tag{2.4}$$

From (2.4) we have

$$Q\xi = (2mk)\,\xi \tag{2.5}$$

where Q denotes the Ricci operator defined by

$$S(X, Y) = g(Q X, Y) \tag{2.6}$$

and S is the Ricci tensor.

3. CONTACT MANIFOLD SATISFYING $R(\xi, X) . \bar{C} = 0$

The first author and N. Guha have considered in their paper [3] Sasakian manifold M^{2m+1} satisfying $R(X, Y) . \bar{C} = 0$. In this paper we have considered the weaker hypothesis $R(\xi, Y) . \bar{C} = 0$ instead of $R(X, Y) . \bar{C} = 0$.

We suppose that

$$R(\xi, X) . \bar{C} = 0. \tag{3.1}$$

We have

$$\bar{C}(X, Y) Z = R(X, Y) Z - \frac{1}{2m-1} [g(Y, Z) QX - g(X, Z) QY + + S(Y, Z) X - S(X, Z) Y] \tag{3.2}$$

where S is the Ricci tensor and Q is defined by (2.6).

Now

$$g(\bar{C}(\xi, Y) . Z, \xi) = \frac{1}{2m-1} [(2m+1) K \eta(Y) \eta(Z) - - k g(Y, Z) - S(Y, Z)]. \tag{3.3}$$

Using (2.4), (2.6) and (3.2) we have

$$(R(\xi, Y) . \bar{C}) (U, V) W = R(\xi, Y) \bar{C}(U, V) W - \bar{C}(R(\xi, Y) U, V) W - - \bar{C}(U, R(\xi, Y) V) W - \bar{C}(U, V) R(\xi, Y) W. \tag{3.4}$$

In virtue of (3.1) we have

$$R(\xi, Y) \bar{C}(U, V) W - \bar{C}(R(\xi, Y) U, V) W - - \bar{C}(U, R(\xi, Y) V) W - \bar{C}(U, V) R(\xi, Y) W = 0. \tag{3.5}$$

This gives

$$g(R(\xi, Y) . \bar{C}(U, V) W, \xi) - g(\bar{C}(R(\xi, Y) U, V) W, \xi) - - g(\bar{C}(U, R(\xi, Y) V) W, \xi) - g(\bar{C}(U, V) R(\xi, Y) W, \xi) = 0. \tag{3.6}$$

Now putting $Y=U=e_i$, where $\{e_i\}$, $i=1, 2, \dots, 2m+1$ be an orthonormal basis of the tangent space at any point of the manifold and taking sum for $1 \leqq i \leqq 2m+1$ of the relation (3.6) we get

$$g(R(\xi, e_i) \bar{C}(e_i, V) W, \xi) - g(\bar{C}(R(\xi, e_i) e_i, V) W, \xi) - - g(\bar{C}(e_i, R(\xi, e_i) V) W, \xi) - g(\bar{C}(e_i, V) R(\xi, e_i) W, \xi) = 0. \tag{3.7}$$

But

$$g(R(\xi, e_i)\ \bar{C}(e_i, V)\, W, \xi) = -\frac{k\,r}{2m-1}\, g(V, W) - k\, g(\bar{C}(\xi, V)\, W, \xi) \quad (3.8)$$

by (2.4), (2.6) and (3.2)

$$g(\bar{C}(R(\xi, e_i)\, e_i, V)\, W, \xi) = 2mk\ g(\bar{C}(\xi, V)\, W, \xi), \quad (3.9)$$

by (2.4) and (3.2)

$$g(\bar{C}(e_i\, R(\xi, e_i)\, V)\, W, \xi) = -\,k\, g(\bar{C}(\xi, V)\, W, \xi), \quad (3.10)$$

by (2.4) and (3.2)

$$g(\bar{C}(e_i, V)\, R(\xi, e_i)\, W, \xi) = -\frac{k\,r}{2m-1}\,\eta(V)\,\eta(W), \quad (3.11)$$

where r denotes the scalar curvature. Now from (3.7) using (3.3), (3.8), (3.9), (3.10) and (3.11) we get

$$K\,[2m(2m+1)\, K\,\eta(V)\,\eta(W) - 2mk\ g(V, W) - 2m\ S(V, W) + $$
$$+\, r\, g(V, W) - r\ \eta(V)\,\eta(W)] = 0.$$

Then either $\mathbf{R} = 0$ or

$$S(V, W) = \left[(2m+1)\ K - \frac{r}{2m}\right]\eta(V)\,\eta(W) + \left(\frac{r}{2m} - K\right) g(V, W) \quad (3.12)$$

in which case this means that the structure is η-Einstein. If $k = 0$, then from (2.4) we get

$$R(X, Y)\ \xi = 0. \quad (3.13)$$

But we know the following results :

Result 1 [2]. Let $M^{2m+1}(\phi, \eta, \xi, g)$ be a contact metric manifold with $R(X, Y)\ \xi = 0$ for all vector fields X, Y. Then M^{2m+1} is locally the Riemannian product of a flat $(m+1)$ - dimensional manifold and m-dimensional manifold of positive curvature 4.

Result 2 [5]. Let M^{2m+1} be an η-Einstein contact metric manifold of dimension $2m + 1 \geqq 5$. If ξ belongs to the k-nullity distribution, then $K = 1$ and the structure is Sasakian.

Result 3 [3]. Let $M^{2m+1}(\phi, \eta, \xi, g)$ be a Sasakian manifold satisfying $R(X, Y) \,.\, \bar{C} = 0$. Then the manifold M^{2m+1} is locally isometric to S^{2m+1} (1).

Hence from the above three results and (3.12), (3.13) we can state the following theorem :

Theorem 1. Let $M^{2m+1}(\phi, \eta, \xi, g)$ be a contact metric manifold with ξ belonging to the K-nullity distribution satisfying $R(\xi, X) . \bar{C} = 0$. Then either M^{2m+1} is locally the Riemannian product of a flat $(m+1)$-dimensional manifold and an m-dimensional manifold of positive curvature 4 or M^{2m+1} is locally isometric to S^{2m+1} (1).

Note : From (3.12) and Result 2 the theorem of [3] follows.

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