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K-CONTACT AND SASAKIAN MANIFOLD WITH CONSERVATIVE PROJECTIVE CURVATURE TENSOR

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Summary : The object of this paper is to study a K-contact and Sasakian manifold with div $P=0$ where P is the projective curvature tensor and 'div' denotes divergence.

KONSERVATİF PROJEKTİF EĞRİLİK TENSÖRÜNE SAHİP K-KONTAKT SASAKİ MANİFOLDU

Özet : Bu çalışmada, P projektif eğrilik tensörünü ve 'div' diverjansı göstermek üzere, div $P = 0$ koşuluna uyan bir K-kontakt Sasaki manifoldu incelenmektedir.

1. INTRODUCTION

Let (M^n, g) be a contact Riemannian manifold with contact form η, the associated vector field ξ, (1 - 1)-tensor field ϕ, and the associated Riemannian metric g. If ξ is a killing vector field, then M^n is called a K-contact Riemannian manifold [1], [3]. A K-contact Riemannian manifold is called Sasakian [1] if and only if

$$(\nabla_X \phi)(Y) = g(X, Y)\,\xi - \eta(Y)\,X \tag{1}$$

holds, where ∇ denotes the operator of covariant differentiation with respect to g. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian. This paper deals with K-contact and Sasakian manifolds in which projective curvature tensor P of type (1,3) is conservative [2], i.e. the divergence of P is zero. It is shown that $P(\xi, X)\,\xi=0$ for every vector field X when such a manifold is K-contact and $P(\xi, X)\;Y=0$ for every X and Y when it is Sasakian. A projective symmetric Riemannian manifold [5], i.e. a manifold in which $\nabla P = 0$, is evidently a manifold in which the divergence of P is zero. Finally, it is shown that $P(X, Y)\,Z = 0$ for all $X, Y, Z,$ when a K-contact Riemannian manifold is projective symmetric.

2. PRELIMINARIES

Let R, S, r denote respectively the curvature tensor of type (1.3), the Ricci tensor of type (0,2) and the scalar curvature of M^n. It is known that in a contact manifold M^n the Riemannian metric may be so chosen that the following relations hold [3], [4] :

$$\phi(\xi) = 0 \tag{2.1}$$

$$\eta(\xi) = 1 \tag{2.2}$$

$$\phi^2 X = -X + \eta(X)\xi \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.4}$$

and

$$g(\xi, X) = \eta(X) \tag{2.5}$$

for any vector fields X, Y.

If M^n is a K-contact Riemannian manifold, then besides (2.1), (2.2), (2.3), (2.4) and (2.5) the following relations hold [1], [3], [4] :

$$\nabla_X \xi = -\phi X \tag{2.6}$$

$$S(X, \xi) = (n-1)\eta(X) \tag{2.7}$$

$$g(R(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y) \tag{2.8}$$

$$R(\xi, X)\xi = -X + \eta(X)\xi \tag{2.9}$$

and

$$(\nabla_X \phi)(Y) = R(\xi, X)Y \tag{2.10}$$

for any vector fields X, Y.

Further, since ξ is a killing vector, S and r remain invariant under it, that is

$$\underset{\xi}{L} S = 0 \tag{2.11}$$

and

$$\underset{\xi}{L} r = 0, \tag{2.12}$$

where L denotes Lie derivation.

3. K-CONTACT RIEMANNIAN MANIFOLD SATISFYING DIV $P = 0$

We have

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]. \tag{3.1}$$

Let us suppose that in a K-contact Riemannian manifold

$$\text{Div } P = 0. \tag{3.2}$$

Then from (3.1) we get

$$(\nabla_U P)(X, Y) Z = (\nabla_U R)(X, Y) Z - \frac{1}{n-1} [(\nabla_U S)(Y, Z) X - - (\nabla_U S)(X, Z) Y]. \tag{3.3}$$

Contraction of (3.3) gives

$$(\text{Div } P)(X, Y) Z = \frac{n-2}{n-1} [(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)]. \tag{3.4}$$

From (3.2) and (3.4) we have

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0. \tag{3.5}$$

From (2.11) we have

$$(\nabla_\xi S)(Y, Z) = - S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi) \tag{3.6}$$

and from (2.12) we have

$$dr(\xi) = 0. \tag{3.7}$$

Putting $X = \xi$ in (3.5) we get

$$(\nabla_\xi S)(Y, Z) - (\nabla_Y S)(\xi, Z) = 0. \tag{3.8}$$

In virtue of (2.6), (2.7), (3.6) and (3.8) we get

$$S(\phi Z, Y) - (n-1) Y \eta(Z) + (n-1) \eta(\nabla_Y Z) = 0. \tag{3.9}$$

Putting $Z = \phi Z$ in (3.9) we get

$$S(\phi^2 Z, Y) - (n-1) Y \eta(\phi Z) + (n-1) \eta(\nabla_Y \phi Z) = 0. \tag{3.10}$$

Since ξ is killing vector, $g(\phi X, \xi)=0$. Hence using (2.3) and (2.7), the equation (3.10) takes the form

$$S(Z, Y) = (n-1) [\eta(Z) \eta(Y) + \eta(\nabla_Y \phi Z)]. \tag{3.11}$$

But $\eta(\nabla_Y \phi Z) = g(\nabla_Y \phi Z, \xi) = g(R(\xi, Y) Z, \xi)$. Therefore (3.11) can be written as

$$S(Z, Y) = (n-1) g(Z, Y) \quad \text{[by (2.8)]}. \tag{3.12}$$

Hence in virtue of (3.12) we get

$$P(X, Y) Z = R(X, Y) Z - \{g(Y, Z) X - g(X, Z) Y\}. \tag{3.13}$$

Putting $X = Z = \xi$ in (3.13), we have

$$\begin{aligned} P(\xi, Y) \xi &= R(\xi, Y) \xi - \{g(Y, \xi) \xi - g(\xi, \xi) Y\} \\ &= 0 \quad \text{[by (2.2), (2.5) and (2.9)]}. \end{aligned} \tag{3.14}$$

Thus $P(\xi, X)\,\xi = 0$ for every X [written X for Y]. Hence from (3.12) and (3.14) we can state the following theorem:

Theorem 1. If in a K-contact Riemannian manifold $M^n\,(n > 2)$ the relation Div $P = 0$ holds, then the manifold is an Einstein manifold and $P(\xi, X)\,\xi = 0$ for every X.

4. SASAKIAN MANIFOLD SATISFYING DIV $P = 0$

We now consider a Sasakian manifold satisfying Div $P = 0$. Since the manifold is Sasakian, from (1) we have

$$(\nabla_X \phi)(Y) = g(X, Y)\,\xi - \eta(Y)\,X.$$

Hence, by (2.10), we get

$$R(\xi, X)\,Y = g(X, Y)\,\xi - \eta(Y)\,X. \tag{4.1}$$

Now by putting $X = \xi$, $Y = X$ and $Z = Y$ in (3.13) we have

$$\begin{aligned} P(\xi, X)\,Y &= R(\xi, X)\,Y - \{g(X, Y)\,\xi - g(\xi, Y)\,X\} = \\ &= R(\xi, X)\,Y - g(X, Y)\,\xi + \eta(Y)\,X = \\ &= g(X, Y)\,\xi - \eta(Y)\,X - g(X, Y)\,\xi + \eta(Y)\,X = 0 \text{ [by (4.1)]}. \end{aligned}$$

Thus $P(\xi, X)\,Y = 0$ for every X, Y. Hence we can state the following theorem:

Theorem 2. If a Sasakian manifold $M^n\,(n > 2)$ satisfies Div $P = 0$, then $P(\xi, X)\,Y = 0$ for every X, Y.

5. PROJECTIVE SYMMETRIC K-CONTACT RIEMANNIAN MANIFOLD

For a projective symmetric Riemannian manifold we have $\nabla P = 0$. Hence Div $P = 0$. Thus in a projective symmetric K-contact Riemannian manifold the relation (3.13) holds. It follows from (3.13) that

$$(\nabla_U P)(X, Y)\,Z = (\nabla_U R)(X, Y)\,Z. \tag{5.1}$$

Since $\nabla P = 0$, it follows from (5.1) that

$$\nabla R = 0. \tag{5.2}$$

That is, the manifold is locally symmetric. But it is known that [6] a locally symmetric K-contact Riemannian manifold has constant curvature 1. Hence

$$R(X, Y)\,Z = g(Y, Z)\,X - g(X, Z)\,Y. \tag{5.3}$$

Therefore from (3.13) it follows that $P(X, Y) Z = 0$ for all X, Y, Z. Hence we can state the following theorem:

Theorem 3. If $M^n (n > 2)$ is a K-contact Riemannian manifold satisfying $\nabla P = 0$, then $P(X, Y) Z = 0$ for every X, Y, Z.

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