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# ON GENERALIZED RICCI 2-RECURRENT RIEMANNIAN MANIFOLD

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**Summary** : The object of this paper is to study a Riemannian manifold called generalized Ricci 2-recurrent Riemannian manifold.

## GENELLEŞTİRİLMİŞ RİCCİ "2-RECURRENT" RIEMANN MANIFOLDU HAKKINDA

**Özet** : Bu çalışmada, "genelleştirilmiş Ricci '2-recurrent' Riemann manifoldu" adı verilen bir Riemann manifoldu incelenmektedir.

**1. Introduction.** A non-flat Riemannian manifold of dimension $n$ is called a generalized 2-recurrent Riemannian manifold [1] if the Riemannian curvature tensor $R$ satisfies the condition

$$(\nabla_V \nabla_U R)(X, Y) Z = A(V)(\nabla_U R)(X, Y) Z + B(U, V) R(X, Y) Z \tag{1.1}$$

where $A$ is a 1-form, $B$ is a non-zero (0,2) tensor and $\nabla$, the Levi-Civita connection of the manifold. Such a manifold has been denoted by $G\{^2K_n\}$. If $A = 0$, the manifold reduces to a 2-recurrent manifold introduced by Lichnerowicz [2] and such a manifold is denoted by $^2K_n$. When the Ricci tensor $S$ satisfies

$$(\nabla_V \nabla_U S)(Y, Z) = A(V)(\nabla_U S)(Y, Z) + B(U, V) S(Y, Z)$$

where $A$ and $B$ are stated earlier, then the manifold is called a generalized Ricci 2-recurrent Riemannian manifold and such a manifold is denoted by $G\{^2R_n\}$. If $A = 0$, then the space reduces to a Ricci 2-recurrent space, studied by Chaki and Roy Chowdhury [3]. Such a manifold is denoted by $^2R_n$.

Obviously every $G\{^2K_n\}$ is a $G\{^2R_n\}$ but the converse is not necessarily true. The question as to when a $G\{^2R_n\}$ can be a $G\{^2K_n\}$ has been considered in section 2 of this paper. In section 3 it is shown that if the scalar curvature $r$ is constant then $r$ must be zero and if the tensor $B$ is symmetric then the vector fields corresponding to the 1-form $A$ and $dr$ are collinear. In the last section $G\{^2R_n\}$ admitting a parallel vector field has been studied.

**2.** It is known that the conformal curvature tensor $C$ of a Riemannian manifold is given by

$$\begin{aligned} C(X, Y, Z, W) = R(X, Y, Z, W) - \frac{1}{n-2}[g(Y, Z)\, S(X, W) - \\ - g(X, Z)\, S(Y, W) + S(Y, Z)\, g(X, W) - \\ - S(X, Z)\, g(Y, W)] + \frac{r}{(n-1)(n-2)}[g(Y, Z)\, g(X, W) - \\ - g(X, Z)\, g(Y, W)] \end{aligned} \tag{2.1}$$

where $S$ is the Ricci tensor and $r$ is the scalar curvature of the manifold.

Now let the Riemannian manifold be a $G\{^2R_n\}$. Then

$$(\nabla_V \nabla_U S)(Y, Z) = A(V)(\nabla_U S)(Y, Z) + B(U, V)\, S(Y, Z). \tag{2.2}$$

From (2.2) we get

$$\nabla_V \nabla_U r = A(V)\, \nabla_U r + B(U, V)\, r. \tag{2.3}$$

By virtue of (2.2) and (2.3) it follows from (2.1) that

$$\begin{aligned} (\nabla_V \nabla_U C)(X, Y, Z, W) = (\nabla_V \nabla_U R)(X, Y, Z, W) + \\ + B(U, V)\, C(X, Y, Z, W) + A(V)(\nabla_U C)(X, Y, Z, W) - \\ - A(V)(\nabla_U R)(X, Y, Z, W) - B(U, V)\, R(X, Y, Z, W) \end{aligned}$$

or

$$\begin{aligned} (\nabla_V \nabla_U C)(X, Y, Z, W) - A(V)(\nabla_U C)(X, Y, Z, W) - \\ - B(U, V)\, C(X, Y, Z, W) = (\nabla_V \nabla_U R)(X, Y, Z, W) - \\ - A(V)(\nabla_U R)(X, Y, Z, W) - B(U, V)\, R(X, Y, Z, W). \end{aligned} \tag{2.4}$$

Conversely, if (2.4) holds, putting $Y = Z = e_i$ in (2.4) where $\{e_i\}$, $i = 1, \ldots, n$ be an orthonormal basis of the tangent space at any point and taking sum over $i$, $1 \leqq i \leqq n$ we get

$$\begin{aligned} (\nabla_V \nabla_U C)(X, W) - A(V)(\nabla_U C)(X, W) - B(U, V)\, C(X, W) = \\ = (\nabla_V \nabla_U S)(X, W) - A(V)(\nabla_U S)(X, W) - B(U, V)\, S(X, W) \end{aligned}$$

which reduces in virtue of $C(X, W) = 0$ to

$$(\nabla_V \nabla_U S)(X, W) = A(V)(\nabla_U S)(X, W) + B(U, V)\, S(X, W). \tag{2.5}$$

From (2.4) and (2.5) we can state the following theorem:

**Theorem 1.** A necessary and sufficient condition that a Riemannian manifold be a $G\{^2R_n\}$ is that (2.4) holds.

In particular, if the Riemannian manifold is conformal to a flat manifold or if $n = 3$ then the conformal curvature tensor $C = 0$. In the first case it follows from (2.4) that the $G\{{}^2R_n\}$ is a $G\{{}^2K_n\}$. In the second case it follows that the $G\{{}^2R_3\}$ is a $G\{{}^2K_3\}$.Thus we have the following theorem:

Theorem 2. Every $G\{{}^2R_n\}$ $(n > 3)$ is a $G\{{}^2K_n\}$ if it is conformal to a flat manifold and every $G\{{}^2R_3\}$ is a $G\{{}^2K_3\}$.

**3.** From (2.3) it follows that if the scalar curvature $r$ is constant then it must be zero.

Again from (2.3) it follows that

$$A(V)\,\nabla_U r - A(U)\,\nabla_V r + [B(U, V) - B(V, U)]\;r = 0. \tag{3.1}$$

If the tensor $B$ is symmetric then we get from (3.1)

$$A(V)\,\nabla_U r - A(U)\,\nabla_V r = 0$$

or

$$A(V)\;dr(U) - A(U)\;dr(V) = 0.$$

From the above discussion we can state the following theorem:

Teorem 3. If the scalar curvature of a $G\{{}^2R_n\}$ is constant, then it must be zero and if the tensor $B$ is symmetric then the vector fields corresponding to the 1-forms $A$ and $dr$ are collinear.

4. $G\{{}^2R_n\}$ **admitting a parallel vector field.** A vector field $Q$ is said to be parallel [4] if

$$\nabla_X Q = 0. \tag{4.1}$$

Then from the definition of

$$R(X, Y)\;Z = \nabla_X\,\nabla_Y\,Z - \nabla_Y\,\nabla_X\,Z - \nabla_{[X,\,Y]}\,Z$$

we get

$$R(X, Y)\;Q = 0 \tag{4.2}$$

and hence

$$S(Y, Q) = 0. \tag{4.3}$$

Taking covariant derivative of (4.2) and then applying Bianchi's identity we get

$$(\nabla_Q\,R)\;(X, Y)\;Z = 0. \tag{4.4}$$

From (4.4) it follows that

$$(\nabla_Q\,S)\;(Y, Z) = 0. \tag{4.5}$$

Also from (4.5) we get

$$\nabla_Q\, r = 0. \tag{4.6}$$

Putting $U = Q$ in (2.3) and applying (4.6) we get

$$B(Q, V)\, r = 0$$

from which it follows that either $B(Q, V) = 0$ or $r = 0$. Hence we can state the following theorem:

**Theorem 4.** If a $G\{{}^2R_n\}$ admits a parallel vector field $Q$ then either $B(Q, V) = 0$ or the scalar curvature vanishes.

## REFERENCES

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