THE WAVE SOLUTIONS OF NON-SYMMETRIC UNIFIED FIELD THEORIES OF EINSTEIN, BONNOR AND SCHRÖDINGER

K.B. LAL and MUSTAKEEM

As a sequel to a former paper of the authors, an attempt has been made in the present paper to obtain the wave solutions of the field equations of non-symmetric unified field theories of Einstein, Bonnor and Schrödinger in the space-time represented by the metric

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 + 2A(z,t)\,dzdt + 2B(x,y)\,dxdy.$$

1. Introduction. Recently Lal and Mustakeem [1] [1)] have investigated the wave solutions of the field equations of general relativity in a space-time, represented by the metric

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 + 2Adzdt + 2Bdxdy\,, \tag{1.1}$$

where $A = A(z,t)$ and $B = B(x,y)$. In this paper we have attempted to obtain the wave solutions of the field equations of non-symmetric unified field theories of Einstein, Bonnor and Schrödinger in the space-time (1.1).

The field equations of A.Einstein's unified field theory [2] are

$$g_{ij;k} \equiv g_{ij,k} - g_{sj}\Gamma^s_{ik} - g_{is}\Gamma^s_{jk} = 0, \tag{1.2}$$

$$\Gamma_t \equiv \Gamma^s_{\underset{\vee}{ts}} = 0\,, \tag{1.3}$$

$$\left.\begin{aligned} &\text{a) } R_{kl} \equiv \Gamma^s_{kl,s} - \Gamma^s_{ks,l} - \Gamma^s_{tl}\Gamma^t_{ks} + \Gamma^s_{ts}\Gamma^t_{kl} = 0 \\ &\text{b) } (i)\ R_{\underline{ij}} = 0\,;\ (ii)\ R_{\underset{\vee}{ij},k} + R_{\underset{\vee}{jk},i} + R_{\underset{\vee}{ki},j} = 0 \end{aligned}\right\} \tag{1.4}$$

where a comma followed by an index denotes ordinary partial differentiation and the Latin indices take the vaules 1, 2, 3, 4. A bar (-) and a hook (v) under two indices denote respectively symmetry and anti-symmetry between them.

Following the above notations the field equations of W.B. Bonnor [3] are given by (1.2), (1,3) and

1) Numbers in brackets refer to the references at the end of the paper.

$$\left.\begin{aligned} &\text{a) } R_{\underline{ij}} + p^2\, U_{\underline{ij}} = 0\,, \\ &\text{b) } (R_{\underset{\vee}{ij},k} + R_{\underset{\vee}{jk},i} + R_{\underset{\vee}{ki},j}) + p^2\, (U_{\underset{\vee}{ij},k} + U_{\underset{\vee}{jk},i} + U_{\underset{\vee}{ki},j}) = 0, \end{aligned}\right\} \tag{1.5}$$

where R_{ij} is the Ricci tensor, p is an arbitrary real or imaginary constant and U_{ik} is given by

$$U_{ik} = g_{\underset{\vee}{ki}} - g^{\overset{mn}{\vee}}\, g_{im}\, g_{nk} + \frac{1}{2}\, g^{\overset{mn}{\vee}}\, g_{nm}\, g_{ik}\,. \tag{1.6}$$

Using similar notations the field equations of E.Schrödinger ([4], [5]) are given by (1.2), (1.3) and

$$\begin{aligned} &\text{a) } R_{\underline{ij}} + \lambda g_{\underline{ij}} = 0\,, \\ &\text{b) } (R_{\underset{\vee}{ij}} + \lambda g_{\underset{\vee}{ij}})_{,k} + (R_{\underset{\vee}{jk}} + \lambda g_{\underset{\vee}{jk}})_{,i} + (R_{\underset{\vee}{ki}} + \lambda g_{\underset{\vee}{ki}})_{,j} = 0 \end{aligned} \tag{1.7}$$

where λ is a non-vanishing constant.

2. Calculation of g_{ij}. We have obtained [1] the transverse-electro-magnetic wave solutions F_{ij} of the generalized Maxwell's equations in the space-time (1.1) which are given by

$$F_{ij} = \begin{bmatrix} 0 & 0 & -\sigma_1 & \sigma_1 \\ 0 & 0 & \rho_1 & -\rho_1 \\ \sigma_1 & -\rho_1 & 0 & 0 \\ -\sigma_1 & \rho_1 & 0 & 0 \end{bmatrix}, \tag{2.1}$$

where ρ_1 and σ_1 are arbitrary functions of x, y and z-t, which satisfy the condition $\partial\rho_1/\partial x + \partial\sigma_1/\partial y = 0$.

In this section we shall determine the non-symmetric g_{ij} corresponding to above F_{ij}. Assume that

$$g_{ij} = g_{\underline{ij}} + g_{\underset{\vee}{ij}} = h_{ij} + f_{ij}\,, \tag{2.2}$$

where $g_{\underline{ij}} = h_{ij}$ is the symmetric part coinciding with the metric tensor of the Riemannian space-time defined by the line element (1.1) and $g_{\underset{\vee}{ij}} = f_{ij}$ is the anti-symmetric part of g_{ij} corresponding to electromagnetic field (2.1). Thus using (1.1) and (2.2) g_{ij} can be put as

$$(g_{ij}) = \begin{bmatrix} -1 & B+f_{12} & f_{13} & f_{14} \\ B-f_{12} & -1 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & -1 & A+f_{34} \\ -f_{14} & -f_{24} & A-f_{34} & 1 \end{bmatrix}. \tag{2.3}$$

To connect the F_{ij} given by (2.1) with g_{ij} we shall use the relation

$$F_{ij} = \frac{1}{2}\varepsilon_{ijkl}\sqrt{-g}\,g^{kl} \quad (g = \det(g_{ij})) \tag{2.4}$$

introduced by Ikeda [6], where g^{ij} is the contravariant tensor of g_{ij} and $\varepsilon_{ijkl} = +1$ or -1 according as i, j, k, l have even or odd permutations. From (2.1) we have $F_{34} = F_{12} = 0$. Hence from (2.4) we get

$$g^{12} = g^{21}\,,\ g^{34} = g^{43}\,. \tag{2.5}$$

Calculating the contravariant components $g^{12}, g^{21}, g^{34}, g^{43}$ from (2.3) and substituting into (2.5) we obtain the following equations :

$$f_{12}(1 + A^2 - f_{34}^2) + f_{34}(f_{13}f_{24} - f_{14}f_{23}) = 0\,,$$
$$(1 - B^2)f_{34} + f_{12}^2 f_{34} - f_{12}(f_{13}f_{24} - f_{14}f_{23}) = 0\,,$$

which on combining are reduced to $(1 - B^2)f_{34}^2 + (1 + A^2)f_{12}^2 = 0$, where $(1 - B^2) > 0$, showing that $f_{12} = f_{34} = 0$. Thus (2.3) becomes

$$(g_{ij}) = \begin{bmatrix} -1 & B & f_{13} & f_{14} \\ B & -1 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & -1 & A \\ -f_{14} & -f_{24} & A & 1 \end{bmatrix}. \tag{2.6}$$

The determinant g formed from the above g_{ij} is given by

$$\begin{aligned} g = &-(1 + A^2)(1 - B^2) - f_{13}^2 + f_{14}^2 - f_{23}^2 + f_{24}^2 + (f_{13}f_{24} - f_{14}f_{23})^2 \\ &+ 2A(f_{13}f_{14} + f_{23}f_{24}) + 2B(f_{14}f_{24} - f_{13}f_{23}) + \\ &+ 2AB(f_{13}f_{24} - f_{14}f_{23})\,. \end{aligned} \tag{2.7}$$

Assuming that $f_{13}f_{24} - f_{14}f_{23} = 0$, we have from (2.6)

$$\begin{aligned} g\,g^{13} &= -g\,g^{31} = -Bf_{23} + ABf_{24} - f_{13} + Af_{14}\,, \\ g\,g^{14} &= -g\,g^{41} = ABf_{23} + Bf_{24} + Af_{13} + f_{14}\,, \\ g\,g^{23} &= -g\,g^{32} = -f_{23} + Af_{24} - Bf_{13} + ABf_{14}\,, \\ g\,g^{24} &= -g\,g^{42} = Af_{23} + f_{24} + ABf_{13} + Bf_{14}\,. \end{aligned} \tag{2.8}$$

Combining (2.1), (2.4) and (2.8) we get the four equations

$-Bf_{23} + ABf_{24} - f_{13} + Af_{14} = \sqrt{-g}\,\rho_1$, etc., which on solving give

$$\begin{aligned} f_{13} &= -f_{14} = \sqrt{-g}\,(1 + A)(\rho_1 - B\sigma_1)\,/\,(1 + A^2)(1 - B^2)\,, \\ f_{23} &= -f_{24} = \sqrt{-g}\,(1 + A)(\sigma_1 - B\rho_1)\,/\,(1 + A^2)(1 - B^2)\,, \end{aligned} \tag{2.9}$$

and we have $g = -(1 + A^2)(1 - B^2)$. Substituting this value of g in (2.9) we have

$$f_{13} = -f_{14} = (1 + A)(\rho_1 - B\sigma_1)/\sqrt{-g} = \rho \text{ (say)},$$
$$f_{23} = -f_{24} = (1 + A)(\sigma_1 - B\rho_1)/\sqrt{-g} = \sigma \text{ (say)},$$

where ρ and σ are functions of $(z\text{-}t)$. Thus finally we have the metric g_{ij} as

$$(g_{ij}) = \begin{bmatrix} -1 & B & \rho & -\rho \\ B & -1 & \sigma & -\sigma \\ -\rho & -\sigma & -1 & A \\ \rho & \sigma & A & 1 \end{bmatrix}. \tag{2.10}$$

3. Connections $\Gamma^k{}_{ij}$ corresponding to g_{ij}. Let us put

$$\Gamma^k{}_{ij} = p^k{}_{ij} + q^k{}_{ij}, \tag{3.1}$$

where $p^k{}_{ij} = p^k{}_{ji} = \Gamma^k{}_{\underline{ij}}$ and $q^k{}_{ij} = -q^k{}_{ji} = \Gamma^k{}_{\overset{}{\underset{\vee}{ij}}}$. For the above substitution the field equation (1.2) will give 64 equations involving 24 q's and 40 p's.

Putting $g_{ij\,;\,k} = E^k{}_{ij}$, we have from (1.2) with $i = j$ the following equations :

$$\begin{aligned}
E^k{}_{11} &: p^1{}_{1k} - B p^2{}_{1k} + \rho q^3{}_{1k} - \rho q^4{}_{1k} = 0\,, \\
E^k{}_{22} &: -B p^1{}_{2k} + p^2{}_{2k} + \sigma q^3{}_{2k} - \sigma q^4{}_{2k} = 0\,, \\
E^k{}_{33} &: -\rho q^1{}_{3k} - \sigma q^2{}_{3k} + p^3{}_{3k} - A p^4{}_{3k} = 0\,, \\
E^k{}_{44} &: \rho q^1{}_{4k} + \sigma q^2{}_{4k} - A\, q^3{}_{4k} - p^4{}_{4k} = 0\,.
\end{aligned} \tag{3.2}$$

Now putting $g_{ij\,;\,k} + g_{ji\,;\,k} = E_{ij}{}^{k+}$, from (1.2) with $i \neq j$ and (2.10), we have

$$\begin{aligned}
E_{12}{}^{k+} &: \frac{\partial B}{\partial x^k} - B(p^1{}_{1k} + p^2{}_{2k}) + p^2{}_{1k} + p^1{}_{2k} + \sigma(q^3{}_{1k} - q^4{}_{1k}) + \\
&\qquad + \rho(q^3{}_{2k} - q^4{}_{2k}) = 0\,, \\
E_{13}{}^{k+} &: p^3{}_{1k} + p^1{}_{3k} - \rho(q^1{}_{1k} - q^3{}_{3k} + q^4{}_{3k}) - \sigma q^2{}_{1k} - \\
&\qquad - A\, q^4{}_{1k} - B\, q^2{}_{3k} = 0\,, \\
E_{14}{}^{k+} &: \rho(q^1{}_{1k} + q^3{}_{4k} - q^4{}_{4k}) - p^4{}_{1k} + p^1{}_{4k} + \sigma q^2{}_{1k} - A\, p^3{}_{1k} \\
&\qquad - B\, p^2{}_{4k} = 0\,, \\
E_{23}{}^{k+} &: \sigma(q^2{}_{2k} - q^3{}_{3k} + q^4{}_{3k}) - p^2{}_{3k} - p^3{}_{2k} + \rho q^1{}_{2k} + A\, p^4{}_{2k} \\
&\qquad + B p^1{}_{3k} = 0\,, \\
E_{24}{}^{k+} &: \sigma(q^2{}_{2k} + q^3{}_{4k} - q^4{}_{4k}) + p^2{}_{4k} - p^4{}_{2k} + \rho q^1{}_{2k} - A\, p^3{}_{2k} \\
&\qquad - B p^1{}_{4k} = 0\,,
\end{aligned} \tag{3.3}$$

$$E_{34}{}^{k+} : \frac{\partial A}{\partial x^k} + \rho(q^1{}_{3k} - q^1{}_{4k}) + \sigma(q^2{}_{3k} - q^2{}_{4k}) + p^3{}_{4k} - p^4{}_{3k} -$$
$$- A(p^3{}_{3k} + p^4{}_{4k}) = 0 .$$

Next putting $g_{ji;k} - g_{ij;k} = E_{ij}{}^{k-}$, we have from (1.2) with $i \neq j$ and (2.10)

$$E_{12}{}^{k-} : - B(q^1{}_{1k} - q^2{}_{2k}) + (q^2{}_{1k} - q^1{}_{2k}) + \sigma(p^3{}_{1k} - p^4{}_{1k}) - \rho(p^3{}_{2k} - p^4{}_{2k}) = 0 ,$$

$$E_{13}{}^{k-} : \frac{\partial \rho}{\partial x^k} - \rho(p^1{}_{1k} + p^3{}_{3k} - p^4{}_{3k}) + q^3{}_{1k} - q^1{}_{3k} - \sigma p^2{}_{1k} -$$
$$- Aq^4{}_{1k} + B q^2{}_{3k} = 0 ,$$

$$E_{14}{}^{k-} : - \frac{\partial \rho}{\partial x^k} - \rho(p^1{}_{1k} - p^3{}_{4k} + p^4{}_{4k}) - q^4{}_{1k} - q^1{}_{4k}$$
$$+ \sigma p^2{}_{1k} - A q^3{}_{1k} + B q^2{}_{4k} = 0 ,$$

$$E_{23}{}^{k-} : \frac{\partial \sigma}{\partial x^k} - \sigma(p^2{}_{2k} + p^3{}_{3k} - p^4{}_{3k}) + q^3{}_{2k} - q^2{}_{3k} - \rho p^1{}_{2k} \tag{3.4}$$
$$- A q^4{}_{2k} + B q^1{}_{3k} = 0 ,$$

$$E_{24}{}^{k-} : - \frac{\partial \sigma}{\partial x^k} - \sigma(p^2{}_{2k} - p^3{}_{4k} + p^4{}_{4k}) - q^2{}_{4k} - q^4{}_{2k} + \rho p^1{}_{2k} -$$
$$- A q^3{}_{2k} + B q^1{}_{4k} = 0 ,$$

$$E_{34}{}^{k-} : \rho(p^1{}_{3k} + p^1{}_{4k}) + \sigma(p^2{}_{3k} + p^2{}_{4k}) - (q^3{}_{4k} + q^4{}_{3k}) -$$
$$- A(q^3{}_{3k} - q^4{}_{4k}) = 0 .$$

Following the methods of H. Takeno, M. Ikeda and S. Abe to solve the above 64 equations we first express all p's in terms of q's and $\{{}^{k}_{ij}\}$ by the formula [7]

$$p^k{}_{ij} = \{{}^{k}_{ij}\} + h^{kl}(q^m{}_{lj} f_{jm} + q^m{}_{lj} f_{im}) , \tag{3.5}$$

where $\{{}^{k}_{ij}\}$ are the Christoffel symbols of the second kind formed from $h_{ij} = \underline{g_{ij}}$ given by (1.1) and h^{kl} is the conjugate contravariant tensor of h_{kl}. The non-vanishing independent components of $\{{}^{k}_{ij}\}$ are

$$\begin{aligned} \{{}^{1}_{11}\} &= - B B_1/(1 - B^2) \equiv B\mu , & \{{}^{3}_{33}\} &= A A_3/(1 + A^2) \equiv A\psi, \\ \{{}^{2}_{11}\} &= - B_1/(1 - B^2) \equiv \mu , & \{{}^{4}_{33}\} &= A_3/(1 + A^2) \equiv \psi , \\ \{{}^{1}_{22}\} &= - B_2/(1 - B^2) \equiv \nu , & \{{}^{3}_{44}\} &= - A_4/(1 + A^2) \equiv - \phi, \\ \{{}^{2}_{22}\} &= - B B_2/(1 - B^2) \equiv B\nu , & \{{}^{4}_{44}\} &= A A_4/(1 + A^2) \equiv A\phi , \end{aligned} \tag{3.6}$$

where B_1, B_2, A_3, A_4 stand for $\partial B/\partial x$, $\partial B/\partial y$, $\partial A/\partial z$, $\partial A/\partial t$ respectively.

Inserting the relevant quantities in (3.5) with the help of (2.10) and (3.6) we can express 40 p's in terms of h_{ij} and $q^k{}_{ij}$ as follows :

$$\begin{aligned}
p^1{}_{11} &= B\mu + 2\,B\rho\,(q^3{}_{12} - q^4{}_{12})\,/\,(1 - B^2)\,,\\
p^2{}_{11} &= \mu + 2\,\rho(q^3{}_{12} - q^4{}_{12})\,/\,(1 - B^2)\,,\\
p^3{}_{11} &= 2\,\rho(q^3{}_{13} - q^4{}_{13} - A\,q^3{}_{14} + A\,q^4{}_{14})\,/\,(1 + A^2)\,,\\
p^4{}_{11} &= -\,2\,\rho(A\,q^3{}_{13} - A\,q^4{}_{13} + q^3{}_{14} - q^4{}_{14})\,/\,(1 + A^2)\,,\\
p^1{}_{12} &= (B\sigma - \rho)\,(q^3{}_{12} - q^4{}_{12})/(1 - B^2),\ p^2{}_{12} = (\sigma - B\rho)\,(q^3{}_{12} - q^4{}_{12})/(1 - B^2),\\
p^3{}_{12} &= (\sigma q^3{}_{13} - \sigma q^4{}_{13} + \rho q^3{}_{23} - \rho q^4{}_{23})\,/\,(1 + A^2) - \qquad (3.7)\\
&\quad - A\,(\sigma q^3{}_{14} - \sigma q^4{}_{14} + \rho q^3{}_{24} - \rho q^4{}_{24})\,/\,(1 + A^2)\,,\\
p^4{}_{12} &= -\,A\,(\sigma q^3{}_{13} - \sigma q^4{}_{13} + \rho q^3{}_{23} - \rho q^4{}_{23})\,/\,(1 + A^2) -\\
&\quad - (\sigma q^3{}_{14} - \sigma\,q^4{}_{14} + \rho q^3{}_{24} - \rho q^4{}_{24})\,/\,(1 + A^2)\,,
\end{aligned}$$

....

and the expressions for $p^k{}_{13}, p^k{}_{14}, p^k{}_{22}, p^k{}_{23}, p^k{}_{24}, p^k{}_{33}, p^k{}_{34}, p^k{}_{44}$ are omitted for brevity's sake.

On substituting the values of p's in terms of q's obtained from (3.7) in (3.2) and (3.3) we find that 40 equations thus obtained are identically satisfied, while on putting the values of p's in (3.4) we have 24 equations in q's as given below :

$$\begin{aligned}
E_{12}{}^{1-} &: q^1{}_{12} - B\,q^2{}_{12} + \rho\sigma\,\{(1 + A)\,(q^3{}_{13} - q^4{}_{13}) + (1 - A)\,(q^3{}_{14} -\\
&\quad - q^4{}_{14})\}\,/\,(1 + A^2) - \rho^2\,\{(1 + A)\,(q^3{}_{23} - q^4{}_{23})\\
&\quad + (1 - A)\,(q^3{}_{24} - q^4{}_{24})\}\,/\,(1 + A^2) = 0\,,\\
E_{12}{}^{2-} &: q^2{}_{12} - B\,q^1{}_{12} + \sigma^2\,\{(1 + A)\,(q^3{}_{13} - q^4{}_{13})\\
&\quad + (1 - A)\,(q^3{}_{14} - q^4{}_{14})\}\,/\,(1 + A^2) - \rho\sigma\{(1 + A)\,(q^3{}_{23} - q^4{}_{23})\\
&\quad + (1 - A)\,(q^3{}_{24} - q^4{}_{24})\}\,/\,(1 + A^2) = 0\,, \qquad (3.8)
\end{aligned}$$

....

(Similar 20 equations are omitted for brevity's sake).

$$\begin{aligned}
E_{34}{}^{3-} &: q^3{}_{34} - A\,q^4{}_{34} + \{(\rho + B\sigma)\,(\rho q^1{}_{13} + \sigma q^2{}_{13} + \rho q^1{}_{14} + \sigma q^2{}_{14})\\
&\quad + (\sigma + B\rho)\,(\rho q^1{}_{23} + \sigma q^2{}_{23} + \rho q^1{}_{24} + \sigma q^2{}_{24})\}\,/\,(1 - B^2) = 0\,,\\
E_{34}{}^{4-} &: q^4{}_{34} + A\,q^3{}_{34} + (\rho + B\sigma)\,(\rho q^1{}_{13} + \sigma q^2{}_{13} + \rho q^1{}_{14} + \sigma q^2{}_{14})\\
&\quad + (\sigma + B\rho)\,(\rho q^1{}_{23} + \sigma q^2{}_{23} + \rho q^1{}_{24} + \sigma q^2{}_{24})\}\,/\,(1 - B^2) = 0\,.
\end{aligned}$$

Thus 24 q's as obtained from 24 simultaneous equations in (3.8) are given by

$$q^1{}_{12} = q^2{}_{12} = q^3{}_{12} = q^4{}_{12} = q^1{}_{34} = q^2{}_{34} = q^3{}_{13} = q^4{}_{13} = q^3{}_{14} =$$
$$= q^4{}_{14} = q^3{}_{23} = q^4{}_{23} = q^3{}_{24} = q^4{}_{24} = q^3{}_{34} = q^4{}_{34} = 0\,, \tag{3.9}$$
$$q^1{}_{13} = -\,q^1{}_{14} = \beta\mu\,, \qquad q^1{}_{23} = -\,q^1{}_{24} = \alpha\nu,$$
$$q^2{}_{13} = -\,q^2{}_{14} = \alpha\mu\,, \qquad q^2{}_{23} = -\,q^2{}_{24} = \beta\nu,$$

where $\alpha = -\,(\rho + B\sigma)\left(-\,B + \frac{2A\rho\sigma}{(1 + A^2)}\right) / \left\{\left(1 + \frac{2A\sigma^2}{(1 + A^2)}\right)\left(1 + \frac{2A\rho^2}{(1 + A^2)}\right) - \left(-\,B + \frac{2A\rho\sigma}{(1 + A^2)}\right)^2\right\}$

and $\beta = (\rho + B\sigma)\left(1 + \frac{2A\rho^2}{(1 + A^2)}\right) / \left\{\left(1 + \frac{2A\sigma^2}{(1 + A^2)}\right)\left(1 + \frac{2A\rho^2}{(1 + A^2)}\right) - \left(-\,B + \frac{2A\rho\sigma}{(1 + A^2)}\right)^2\right\}.$

Then putting the values of q's from (3.9) in (3.7) we get the values of p's in terms of A, B and ρ,σ as follows :

$$\begin{aligned}
&p^k{}_{11}\ [\,B\mu \quad \mu \quad 0 \quad 0\,],\\
&p^k{}_{12}\ [\,0 \quad 0 \quad 0 \quad 0\,],\\
&p^k{}_{22}\ [\,\nu \quad B\nu \quad 0 \quad 0\,],\\
&p^k{}_{34}\ [\,0 \quad 0 \quad 0 \quad 0\,],\\
&p^k{}_{13}\left[0 \quad 0 \quad \frac{-\,(1 + A)\,M\mu}{(1 + A^2)} \quad \frac{-\,(1 - A)\,M\mu}{(1 + A^2)}\right],\\
&p^k{}_{14}\left[0 \quad 0 \quad \frac{(1 + A)\,M\mu}{(1 + A^2)} \quad \frac{(1 - A)\,M\mu}{(1 + A^2)}\right],\\
&p^k{}_{23}\left[0 \quad 0 \quad \frac{-\,(i + A)\,L\nu}{(1 + A^2)} \quad \frac{-\,(1 - A)\,L\nu}{(1 + A^2)}\right],\\
&p^k{}_{24}\left[0 \quad 0 \quad \frac{(1 + A)\,L\nu}{(1 + A^2)} \quad \frac{(1 - A)\,L\nu}{(1 + A^2)}\right],\\
&p^k{}_{33}\left[\frac{2\,(M\mu + BL\nu)}{(1 - B^2)} \quad \frac{2\,(BM\mu + L\nu)}{(1 - B^2)} \quad A\psi \quad \psi\right],\\
&p^k{}_{44}\left[\frac{2\,(M\mu + BL\nu)}{(1 - B^2)} \quad \frac{2\,(BM\mu + L\nu)}{(1 - B^2)} \quad -\phi \quad A\phi\right],
\end{aligned} \tag{3.10}$$

where $(\rho\alpha + \sigma\beta) = L$ and $(\sigma\alpha + \rho\beta) = M$.

Substituting the values of p's and q's from (3.10) and (3.9) respectively into (3.1) we get the non-symmetric connections as follows :

$$
\begin{array}{llllll}
\Gamma^k_{11} & [B\mu & \mu & 0 & 0 &]\,, \\
\Gamma^k_{22} & [\nu & B\nu & 0 & 0 &]\,, \\
\Gamma^k_{\substack{12\\21}} & [0 & 0 & 0 & 0 &]\,, \\
\Gamma^k_{\substack{34\\43}} & [0 & 0 & 0 & 0 &]\,, \\
\Gamma^k_{13} & \Big[\beta\mu & \alpha\mu & \dfrac{-(1+A)\,M\mu}{(1+A^2)} & \dfrac{-(1-A)\,M\mu}{(1+A^2)} & \Big]\,, \\
\Gamma^k_{14} & \Big[-\beta\mu & -\alpha\mu & \dfrac{(1+A)\,M\mu}{(1+A^2)} & \dfrac{(1-A)\,M\mu}{(1+A^2)} & \Big]\,, \\
\Gamma^k_{23} & \Big[\alpha\nu & \beta\nu & \dfrac{-(1+A)\,L\nu}{(1+A^2)} & \dfrac{-(1-A)\,L\nu}{(1+A^2)} & \Big]\,, \\
\Gamma^k_{24} & \Big[-\alpha\nu & -\beta\nu & \dfrac{(1+A)\,L\nu}{(1+A^2)} & \dfrac{(1-A)\,L\nu}{(1+A^2)} & \Big]\,, \\
\Gamma^k_{31} & \Big[-\beta\mu & -\alpha\mu & \dfrac{-(1+A)\,M\mu}{(1+A^2)} & \dfrac{-(1-A)\,M\mu}{(1+A^2)} & \Big]\,, \\
\Gamma^k_{32} & \Big[-\alpha\nu & -\beta\nu & \dfrac{-(1+A)\,L\nu}{(1+A^2)} & \dfrac{-(1-A)\,L\nu}{(1+A^2)} & \Big]\,, \\
\Gamma^k_{41} & \Big[\beta\mu & \alpha\mu & \dfrac{(1+A)\,M\mu}{(1+A^2)} & \dfrac{(1-A)\,M\mu}{(1+A^2)} & \Big]\,, \\
\Gamma^k_{42} & \Big[\alpha\nu & \beta\nu & \dfrac{(1+A)\,L\nu}{(1+A^2)} & \dfrac{(1-A)\,L\nu}{(1+A^2)} & \Big]\,, \\
\Gamma^k_{33} & \Big[\dfrac{2\,(M\mu+BL\nu)}{(1-B^2)} & \dfrac{2\,(BM\mu+L\nu)}{(1-B^2)} & A\psi & \psi & \Big]\,, \\
\Gamma^k_{44} & \Big[\dfrac{2\,(M\mu+BL\nu)}{(1-B^2)} & \dfrac{2\,(BM\mu+L\nu)}{(1-B^2)} & -\phi & A\phi & \Big]\,.
\end{array}
\tag{3.11}
$$

4. Solution of the Secoud Field Equation *(1.3)*. Inserting relevant quantities in (1.3) we find that out of the four equations $\Gamma^s_{\underset{\vee}{ts}} = 0$ equations for $t = 1$ and $t = 2$ are identically satisfied while equations for $t = 3$ and $t = 4$ are satisfied when the following relation holds :

$$\mu + \nu = 0\ . \tag{4.1}$$

Thus (4.1) is a necessary condition in order that the Einstein's second field equation in the space-time (1.1) be satisfied.

5. Solutions of the Field Equations (1.4). To solve the field equations (1.4) we shall first calculate the components of the generalized Ricci tensor by the formula

$$R_{ij} = \Gamma^{s}{}_{ij,s} - \Gamma^{s}{}_{is,j} - \Gamma^{s}{}_{tj}\Gamma^{t}{}_{is} + \Gamma^{s}{}_{ts}\Gamma^{t}{}_{ij}\,. \tag{5.1}$$

Substituting the values of $\Gamma^{k}{}_{ij}$ from (3.11) into (5.1) we get the following components of the generalized Ricci tensor :

$$R_{11} = \mu_2 + B\mu\nu + \frac{2A}{(1+A^2)}(M\mu)_1 - \frac{4A^2M^2\mu^2}{(1+A^2)^2} - \frac{2A\mu\,(BM\mu + L\nu)}{(1+A^2)}\,,$$

$$R_{22} = \nu_1 + B\mu\nu + \frac{2A}{(1+A^2)}(L\nu)_2 - \frac{4A^2L^2\nu^2}{(1+A^2)^2} - \frac{2A\nu\,(M\mu + BL\nu)}{(1+A^2)}\,,$$

$$R_{12} = -(B\mu)_2 - \mu\nu + \frac{2A}{(1+A^2)}(M\mu)_2 - \frac{4A^2LM\mu\nu}{(1+A^2)^2}\,,$$

$$R_{21} = -(B\nu)_1 - \mu\nu + \frac{2A}{(1+A^2)}(L\nu)_1 - \frac{4A^2LM\mu\nu}{(1+A^2)^2}\,,$$

$$R_{13} = (\beta\mu)_1 + (\alpha\mu)_2 - (1-B)\,\alpha\mu\nu - MN\mu - \left(\frac{(1-A)M\mu}{(1+A^2)}\right)_3 - \left(\frac{(1-A)\,M\mu}{(1+A^2)}\right)_4,$$

$$\begin{aligned}
R_{31} &= -(\beta\mu)_1 - (\alpha\mu)_2 + (1-B)\,\alpha\mu\nu - \\
&\quad - MN\mu - \left(\frac{(1+A)\,M\mu}{(1+A^2)}\right)_3 - \left(\frac{(1-A)\,M\mu}{(1+A^2)}\right)_4,
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
R_{23} &= (\alpha\nu)_1 + (\beta\nu)_2 - (1-B)\,\alpha\mu\nu - \\
&\quad - LN\nu - \left(\frac{(1-A)\,L\nu}{(1+A^2)}\right)_3 - \left(\frac{(1-A)\,L\nu}{(1+A^2)}\right)_4,
\end{aligned}$$

$$\begin{aligned}
R_{24} &= -(\alpha\nu)_1 - (\beta\nu)_2 + (1-B)\,\alpha\mu\nu + \\
&\quad + LS\nu + \left(\frac{(1+A)\,L\nu}{(1+A^2)}\right)_3 + \left(\frac{(1+A)\,L\nu}{(1+A^2)}\right)_4,
\end{aligned}$$

$$R_{14} = -(\beta\mu)_1 - (\alpha\mu)_2 + (1-B)\,\alpha\mu\nu + MS\mu + Q\,,$$

$$R_{41} = (\beta\mu)_1 + (\alpha\mu)_2 - (1-B)\,\alpha\mu\nu + MS\mu + Q\,,$$

$$R_{32} = -(\alpha\nu)_1 - (\beta\nu)_2 + (1-B)\,\alpha\mu\nu - LN\nu - W\,,$$

$$R_{42} = (\alpha\nu)_1 + (\beta\nu)_2 - (1-B)\,\alpha\mu\nu + LS\nu + W\,,$$

$$R_{34} + (A\psi)_4 = R_{43} + (A\phi)_3 = \phi\psi - I - AH\,,$$

$$R_{33} = \psi_4 + A\phi\psi + I + H + 2T,$$
$$R_{44} = -\phi_3 - A\phi\psi + I - H + 2T,$$

where

$$N = \frac{A(1-A)\phi + (1+A)\psi}{(1+A^2)}, \quad S = \frac{A(1+A)\psi - (1-A)\phi}{(1+A^2)},$$

$$Q = \left(\frac{(1+A)\,M\mu}{(1+A^2)}\right)_3 + \left(\frac{(1+A)\,M\mu}{(1+A^2)}\right)_4, \quad W = \left(\frac{(1+A)\,L\nu}{(1+A^2)}\right)_3 + \left(\frac{(1-A)\,L\nu}{(1+A^2)}\right)_4,$$

$$I = \beta^2(\mu^2+\nu^2) + 2\alpha^2\mu\nu, \quad H = \frac{4\,(M^2\mu^2 + 2BML\,\mu\nu + L^2\nu^2)}{(1-B^2)\,(1+A^2)},$$

$$T = \frac{B\{M\mu^2 + (L+M)\,B\,\mu\nu + L\nu^2\}}{(1-B^2)} + \left(\frac{M\mu + BL\nu}{(1-B^2)}\right)_1 + \left(\frac{BM\mu + L\nu}{(1-B^2)}\right)_2 .$$

Now substituting the values of R_{ij} from (5.2) into (1.4a) and using (4.1), we find that it is satisfied if and only if

$$\rho\alpha + \sigma\beta = 0, \quad \sigma\alpha + \rho\beta = 0, \; A_{34} - AA_3A_4\,/\,(1+A^2) = 0, \tag{5.3}$$
$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 0 . \tag{5.4}$$

Next putting R_{ij} from (5.2) into (1.4b) (*i*) and using (4.1) we see that it is satisfied if the relation (5.3) holds, while equation (1.4b) (*ii*), for the same values of R_{ij}, is satisfied if and only if

$$(\beta_3+\beta_4)\,\{(B_{11}-B_{22}) + 2B\,(B_1^{\,2}-B_2^{\,2})\,/\,(1-B^2)\} + B_1\,(\beta_{13}+\beta_{14}+\alpha_{23}+\alpha_{24}) - B_2\,(\alpha_{13}+\alpha_{14}+\beta_{23}+\beta_{24}) = 0\,, \tag{5.5}$$
$$\alpha(\nu_{11}+\nu_{22}) + 2\,(\alpha_1+\beta_2)\,\nu_1 + (\alpha_{11}+\alpha_{22})\,\nu + 2\,(\alpha_2+\beta_1)\,\nu_2 + 2\beta\nu_{21} + 2\beta_{21}\nu + \alpha(\mu+2B\nu)\,(\nu_2-\nu_1) + \nu(\mu+B\nu)\,(\alpha_2-\alpha_1) + \alpha\nu\,\{(B_2-B_1)\,\nu + (\mu_2-\mu_1)\} = 0.$$

Thus, the values of g_{ij} given by (2.10) represent the wave solution of the strong field equation (1.4a) under the conditions (5.3) and (5.4). While the same g_{ij} are the solutions of the weak field equations (1.4b) (*i*) and (1.4b) (*ii*) under the conditions (5.3) and (5.5) respectively.

6. Solutions of the Field Equations of Bonnor. The first two of the field equations of W.B. Bonnor's non-symmetric unified field theory [3] are the same as the field equations (1.2) and (1.3) of Einstein's unified field theory. Hence their solutions in the space-time (1.1) shall be given by (2.10) and (4.1).

To find the solutions of the remaining two field equations, namely (1.5a) and (1.5b), we use the components of R_{ij}, the generalized Ricci tensor given by (5.2). The contravariant components g^{ij} of the tensor g_{ij} given by (2.10) can be shown to be as follows :

$$(g^{ij}) = \frac{1}{m}\begin{bmatrix} -(1+A^2+2\sigma^2 A) & -(1+A^2)B+2A\rho\sigma & (1+A)(B\sigma+\rho) & (1-A)(B\sigma+\rho) \\ -(1+A^2)B+2A\rho\sigma & -(1+A^2+2A\rho^2) & (1+A)(B\rho+\sigma) & (1-A)(B\rho+\sigma) \\ -(1+A)(B\sigma+\rho) & -(1+A)(B\rho+\sigma) & -(1-B^2)+P & (1-B^2)A+P \\ -(1-A)(B\sigma+\rho) & -(1-A)(B\rho+\sigma) & (1-B^2)A+P & (1-B^2)+P \end{bmatrix} \tag{6.1}$$

where $m = (1+A^2)(1-B^2)$ and $P = 2B\rho\sigma + \rho^2 + \sigma^2$.

Then substituting g_{ij} and g^{ij} from (2.10) and (6.1) into (1.6) we get the following components of U_{ik} :

$$\begin{aligned} U_{11} &= \frac{2A}{m}\{2B\rho(B\rho+\sigma)+(\sigma^2-\rho^2)\}, \\ U_{12} &= U_{21} = -\frac{2A}{m}\{B(\rho^2+\sigma^2)+2\rho\sigma\}, \\ U_{22} &= \frac{2A}{m}\{2B\sigma(B\sigma+\rho)+(\rho^2-\sigma^2)\}, \\ U_{13} &= -U_{31} = -U_{14} = U_{41} = -2\rho, \\ U_{23} &= -U_{32} = -U_{24} = U_{42} = -2\sigma, \\ U_{34} &= U_{43} = \frac{2P}{m}, \\ U_{33} &= -\frac{2(1-A+A^2)P}{m}, \quad U_{44} = -\frac{2(1+A+A^2)P}{m}. \end{aligned} \tag{6.2}$$

Putting R_{ij} from (5.2) and U_{ik} from (6.2) into the field equation (1.5a) we find that it is satisfied if and only if

$$m A_{34} - (1-B^2) A A_3 A_4 + (3+2A)(1+A^2)(\rho^2+\sigma^2)(1-B^2-2B^4)p^2 = 0,$$

$$L = 0, \quad M = 0. \tag{6.3}$$

Using (5.2) and (6.2) into the field equation (1.5b), we arrive at the same equation (5.5) of section 5. Therefore, (1.5b) is satisfied if and only if (5.5) is satisfied.

Thus the wave solutions of the field equations of Bonnor's non-symmetric unified field theory for the space-time (1.1) are given by (2.10) under the conditions (4.1), (5.5) and (6.3).

7. Solution of Schrödinger's Field Equations. Here again the first two field equations of E. Schrödinger's non-symmetric unified field theory are same as the first two field equations of Einstein's or Bonnor's unified field equations, given in Section 1, which will again have the solution (2.10) under the condition (4.1). The remaining two field equations are (1.7a) and (1.7b).

On substituting from (5.2) and (2.10) into (1.7a) for $ij = 11, 22, 12$ etc. we find that it is satisfied if and only if

$$\begin{aligned} &A A_3 A_4 - A_{34}(1 + A^2) + \lambda(1 + A^2)^2 = 0, \quad L = 0, \quad M = 0, \\ &B^2 B_1 B_2 + B B_{12}(1 - B^2) + \lambda B(1 - B^2)^2 = 0 \; , \end{aligned} \tag{7.1}$$

while the field equation (1.7b) is satisfied if (5.5) holds.

Thus the wave solutions of the field equations of Schrödinger's non-symmetric unified field theory for the space-time (1.1) are given by (2.10) under the conditions (4.1), (5.5) and (7.1).

REFERENCES

[1] LAL, K.B. and MUSTAKEEM : On wave solutions of the field equations of general relativity, Rendi-Conti, Vol. LVIII, No.1, Serie VIII (1975), 18-23.

[2] EINSTEIN, A. : The meaning of relativity, Fifth ed. Revised, London, Appendix II (1951).

[3] BONNOR, W.B. : The equations of motion in the non-symmetric unified field theory, Proc. of Royal Society, London, A **226** (1954), 266-377.

[4] TAKENO, H. : On some generalized plane wave solutions of non-symmetric unified field theories, II. Tensor, N.S. **8** (1958), 71-78.

[5] SCHRÖDINGER, E. : Space-time structure, Cambridge (1950).

[6] IKEDA, M. : On static solutions of Einstein's generalized theory of gravitation, I, Prog. Theor. Phys. **12** (1954), 17-30.

[7] TAKENO, H., IKEDA, M. and ABE, S. : On solutions of new field equations of Einstein and those of Schrödinger, Prog. Theor. Phys. **6** (1951), 837-848.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GORAKHPUR
GORAKHPUR-273001 INDIA

ÖZET

Yazarların daha önceki bir çalışmalarının devamı olarak, bu çalışmada

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 + 2A(z, t)\, dzdt + 2B(x,y)\, dxdy$$

metriği ile verilen uzay-zamandaki Einstein, Bonnor ve Schrödinger'in non-simetrik birleşik alan teorilerine ait alan denklemlerinin dalga çözümlerini bulma problemiyle uğraşılmıştır.