AN INTERIOR SOLUTION FOR A NEUTRON STAR MODEL IN GENERAL RELATIVITY

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This paper describes the solutions of the field equations (spherically symmetric) of general relativity for neutron star models of constant gravitational mass density. Physical interpretations of the results have also been mentioned where necessary.

1. Introduction. Much progress has been made towards the study of the structure of static fluid spheres (including massive spheres) [1–6] within the framework of general relativity. Considerable attention has also been paid to the study of equilibrium models for zero-temperature stars with very high density, the so called neutron stars [7]. This study is of academic interest while considering problems on gravitational collapse. For instance, one might find upper limit M_{max} for the mass of an equilibrium zero-temperature star of any density. In that case, for a star which manages to lose all its angular momentum while retaining a mass in excess of M_{max}, one has to consider gravitational collapse into relativistic regimes. Planetary structures in general relativity, particularly of Mars, has recently been discussed by the author. To deal with such types of problems one usually recalls the well known Schwarzschild interior or exterior solutions which represent the field of a fluid sphere of constant density ρ.

The following is merely a short review pertaining to the equilibrium models for neutron stars of constant gravitational mass density $H = \rho c^2 + 3p$. As Whittaker [8] has pointed out, it is this expression, rather than ρ, which governs the gravitational attraction of matter. The equation of state is assumed to be [9] $p = q\rho$, where p and ρ are the pressure and energy density respectively and q is a constant. The value of q is $1/3$ for neutron star models, but it may be close to $1/13$ when baryons are present. Recently, in one [10] of his papers [10–13] the author has made use of the concept of constant gravitational mass density to study polytropic fluid spheres in general relativity.

II. Equations of Hydrostatic Equilibrium and the Desired Solutions. In a static spherical coordinate system (ct, r, θ, ϕ) the space-time metric has the standard form :

$$ds^2 = a(r)\,dr^2 + r^2\,d\Omega^2 - b(r)\,c^2\,dt^2 \tag{1}$$

and the time-independent gravitational equations reduce to

$$\frac{1}{abr}\frac{db}{dr} - \frac{1}{r^2}\left(1 - \frac{1}{a}\right) + \lambda = Kp\,, \tag{2}$$

$$\frac{1}{a^2\,r}\frac{da}{dr} + \frac{1}{r^2}\left(1 - \frac{1}{a}\right) - \lambda = K\rho c^2\,, \tag{3}$$

where λ denotes the cosmical constant, the constant K is connected with the gravitational constant G by the equation $Kc^2 = 8\pi\,G/c^2 = 1.87 \times 10^{-27}$ cm gm^{-1},$d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$, and other symbols have their usual meanings. The equations of hydrostatic equilibrium in these coordinates are

$$2\frac{dp}{dr} + (\rho c^2 + p)\frac{1}{b}\frac{db}{dr} = 0\,, \tag{4}$$

and

$$M(r) = \int_0^r 4\pi y^2\,\rho(y)\,dy \equiv \frac{4}{3}\pi r^3 < \rho(r) >\,, \tag{5}$$

where $M(r)$ is the total mass interior to r.

Case (i). According to our assumptions

$$\rho c^2 + 3p = H = \text{constant}\,;\, p = \frac{1}{3}\rho\,, \tag{6}$$

hence, the solution of (4) is obtained as

$$H - \frac{2}{3}\rho = \alpha b\,, \tag{7}$$

where α is the constant of integration. Addition of (2) and (3) would give

$$\frac{1}{b}\frac{db}{dr} + \frac{1}{a}\frac{da}{dr} = K(p + \rho c^2)\,ar\,. \tag{8}$$

Making use of equations (6) and (7) in (8) and integrating, we obtain, after some simplifications,

$$ab = \frac{\beta}{1 - \delta^2 r^2}; \quad \delta = \left(\frac{K\alpha\beta}{2}\right)^{1/2}, \tag{9}$$

where β is another constant of integration. Using the results of (7) and (9), we obtain from (2):

$$(1 - \delta^2 r^2)\, r \frac{db}{dr} + b = \beta + \frac{\beta}{2}(KH - 2\lambda)\, r^2, \tag{10}$$

the solution of which is given by

$$b(r) = \beta \left\{1 + L - \frac{L}{\delta r}(1 - \delta^2 r^2)^{1/2} \sin^{-1} \delta r\right\} + \frac{M}{r}(1 - \delta^2 r^2)^{1/2}, \tag{11}$$

where M and $L \left(= \frac{1}{2\delta^2}(KH - 2\lambda)\right)$ are constants. In order that $b(r)$ be finite at the centre $r = 0$, M must vanish.

Hence the foregoing expression becomes

$$b(r) = \beta \left\{1 + L - \frac{L}{\delta r}(1 - \delta^2 r^2)^{1/2} \sin^{-1} \delta r\right\}. \tag{11$'$}$$

By virtue of this equation, (9) reduces to

$$a(r) = \left\{(1 + L)(1 - \delta^2 r^2) - L(\delta r)^{-1} \frac{(1 - \delta^2 r^2)^{1/2}}{r} \sin^{-1} \delta r\right\}^{-1}. \tag{12}$$

With the help of equations (7), (11′) and the second relation in (9), we obtain the expression for the density at a point distance r from the centre of the neutron star, in the form

$$K\rho(r) = 3\left\{(\lambda - \delta^2) + L\delta \frac{(1 - \delta^2 r^2)}{r} \sin^{-1} \delta r\right\}. \tag{13}$$

(a) **Central density and pressure.** Equation (13) yields the relation

$$(K\rho)_0 = 3\{\lambda + \delta^2 (L - 1)\} = \frac{3}{2} KH - 3\delta^2. \tag{14}$$

From the first equation in (6) and (14), we get the central value of the pressure:

$$(Kp)_0 = \frac{KH}{3} - \frac{1}{3}(K\rho)_0 c^2 = \delta^2 c^2 - \frac{KH}{6}(3c^2 - 2), \tag{15}$$

where the suffix 0 refers to the value evaluated at the centre. Equations (14) and (15) make it clear that for non-negative central values of pressure and density, we must have [1]

$$2\delta^2 \leq KH \leq \frac{6\delta^2 c^2}{3c^2 - 2}. \tag{16}$$

(b) **Fitting up the solution with Schwarzschild's exterior solution.** We further observe that the gradient of p is negative which means that the pressure decreases as the radius of the star increases, and tends to zero as the boundary $r = r_1$ is reached, where $x_1 = \delta r_1$ is the root of the equation

$$\{3c^2 (\lambda - \delta^2) - KH\} x_1 = 3c^2 \left\{\left(\frac{1}{2} KH - \lambda\right) (1 - x_1^2)^{1/2} \sin^{-1} x_1\right\}. \tag{17}$$

At this point we may replace our interior solution by the Schwarzschild's exterior solution

$$b(r) = \frac{1}{a(r)} = 1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2. \tag{18}$$

Further, for the exterior solution $a(r)$ and $b(r)$ must be continuous, and $ab = 1$. In view of this, we obtain from (7) (on putting $p = 0$ under the conditions stated below eqn. (16))

$$1 - \frac{2m}{r_1^2} - \frac{1}{3} \lambda r_1^2 = \frac{H}{\alpha} = \frac{HK\beta}{2\delta^2}, \tag{19}$$

$$\beta + \delta^2 r^2 = 1 \tag{20}$$

respectively. Equations (19) and (20) determine the value of m:

$$m = \frac{1}{12\delta^2} [r^4 (3HK - 2\lambda) \delta^2 - 3r_1^2 (HK - 2\delta^2)]. \tag{21}$$

Physical interpretation of the parameter $HK/2\delta^2$, in terms of central pressure and density can be given by the following equation

$$\frac{HK}{2\delta^2} = 6 \cdot \frac{3 (Kp)_0 + (K\rho)_0 c^2}{18 (Kp)_0 - 4 (K\rho)_0 \left(1 - \frac{3}{2} c^2\right)}. \tag{22}$$

Clearly, m can remain positive only when

[1] This may be taken as an improvement over the author's previous results $2\delta^2 \leq KH \leq 3\delta^2$ (communicated) as obtained for a fluid sphere without introducing in (6) the specified equation of state $p = 1/3\, \rho$.

$$\frac{1}{3}\lambda r^2 + \frac{HK}{2\delta^2}(1 - \delta^2 r_1^2) \leqq 1 . \tag{23}$$

(c) **The speed of sound at the centre.** As an example of the physical application of our results contained in (14) and (15), we may obtain the ratio of the speed of sound to that of light, at the centre of the neutron star, as follows :

Using the equation of state ($p = \rho/3$) and the energy density relation $\varepsilon = \rho c^2$, we get an expression v_s^2/c^4 for the ratio of the speed of sound v_s to that of light c :

$$\frac{v_s^2}{c^4} = \frac{qp}{q\varepsilon} = \frac{1}{c^2}\frac{p}{\rho} . \tag{24}$$

In terms of central density and pressure this becomes

$$\left(\frac{v_s^2}{c^2}\right)_0 = \frac{1}{3}\frac{KH(2-3c^2)+6\delta^2 c^2}{3\,KH - 6\delta^2} . \tag{25}$$

Obviously, therefore,

$$\text{for } [KH]_{\min} = 2\delta^2, \left(\frac{v_s^2}{c^2}\right)_0 \to \infty \qquad \text{(a)}$$

and (26)

$$\text{for } [KH]_{\min} = \frac{6\delta^2 c^2}{3c^2 - 2}, \left(\frac{v_s^2}{c^2}\right)_0 \to 0 . \qquad \text{(b)}$$

Case (ii). $p = \rho / 13$. This does not require a detailed discussion, and therefore, we shall confine ourselves to point out only some modified equations of the foregoing case (*i*). In this case, equations, corresponding to (7), (13), (14), (15), (16), (17), (22), are given, respectively, by

$$G - \frac{2}{13}\rho = lb , \tag{7'}$$

$$K\rho(r) = 13\{(\lambda - \delta^2) + L\,\delta^2(\delta r)^{-1}(1 - \delta^2 r^2)^{1/2}\sin^{-1}\delta r\} , \tag{13'}$$

$$(K\rho)_0 = 13\{\lambda + \delta^2(L-1)\} = 13\left(\frac{KH}{2} - \delta^2\right) , \tag{14'}$$

$$(Kp)_0 = \frac{1}{3}\left\{13\delta^2 c^2 - KH\left(\frac{13}{2}c^2 - 1\right)\right\} , \tag{15'}$$

$$2\delta^2 \leqq KH \leqq \frac{26\,\delta^2 c^2}{13c^2 - 2} , \tag{16'}$$

$$\{13c^2(\lambda - \delta^2) - KH\}\, x_1 = 13c^2 \left\{\left(\frac{1}{2} KH - \lambda\right)(1 - x_1^2)^{1/2} \sin^{-1} x_1\right\} \quad (17')$$

and

$$\frac{HK}{2\delta^2} = \frac{3(Kp)_0 + (Kp)_0\, c^2}{39\,(Kp)_0 - 2(Kp)_0 \left(1 - \frac{13}{2} c^2\right)} . \quad (22')$$

The expression for the velocity of sound, at the centre, similar to (25), is obtained as

$$\left(\frac{v_s^2}{c^2}\right)_0 = \frac{1}{3} \frac{KH(2 - 13c^2) + 26\, \delta^2 c^2}{13\,(KH - 2\delta^2)} \quad (25')$$

and we find that the two limiting values for the ratio $(v_s^2 / c^2)_0$ -one tending to infinity and the other to zero (as in 26 (a) and 26 (b))- with restriction to the inequality (30), again occur, for the simple reason that we have used a similar equation of state ($p = \rho / 13$) under adiabatic approximations.

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REFERENCES

[1] IBEN, ICKO : Jr. Ap. J. **138** (1963), 1090.

[2] HOYLE, F. and FOWLER, W.A. : Mon. Not. Roy. Astr. Soc. **169** (1963), 125; Nature, **197** (1963), 533.

[3] HOYLE, F., FOWLER, W.A., BURBIDGE, G.R. and BURBIDGE, E.M. : Ap. J. **139** (1964), 909.

[4] FOWLER, W.A. : Rev. Mod. Phys. **36** (1964), 545.

[5] TOOPER, ROBERT F. : Ap. J. **140** (1964), 434; Ap. J. **142** (1965), 1541.

[6] SATO, HUMITAKA : Prog. Theo. Phys. **35** (2) (1966), 241.

[7] SALPETER, E.E. : Quasi-Stellar Sources and Gravitational Collapse (Edited by Ivor Robinson, Alfred Schild and E.L. Schucking, Univ. of Chicago Press, (1965), Chap. 32, p.393; Hong-Yee Chiu (Ibid.) ,Chap. 33.

[8] WHITTAKER, E.T. : Proc. Roy. Soc., **149**(A) (1935), 384.

[9] YABUSCHITA, S. : Mon. Not. Roy. Ast. Soc., **167** (1974), 95.

[10] SHARMA, J.P. : Indian J. Phys., **48** (4), (1974).

[11] SHARMA, J.P. : Pure Appl. Geophys., 97 (1972), 14.

[12] SHARMA, J.P. : J. Geophys. Res., **78** (23) (1973), 31.

[13] SHARMA, J.P. : Indian J. Pure Appl., Maths. (in press).

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ÖZET

Bu çalışmada sabit gravitasyon kütlesi yoğunluklu nötron yıldızları modellerinin genel relativitedeki (küresel simetriyi haiz) alan denklemleri anlatılmakta ve gerekli olan yerlerde sonuçların fiziksel yorumları verilmektedir.