PROJECTIVE AFFINE MOTION IN A PRF_n-SPACE, II

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The possibility of the existence of an infinitesimal special projective affine motion in a PRF_n-space is studied.

1. Introduction. Let us consider an n-dimensional affinely connected Finsler space F_n[1] [1]) equipped with $2n$ line elements $(x^i, \dot{x}^i)$ $(i = 1,2,\ldots,n)$ and a fundamental metric function $F(x, \dot{x})$ which is positively homogeneous of degree one in its directional arguments. The fundamental metric tensor of the space is given by

$$g_{ij}(x, \dot{x}) \stackrel{\text{def.}}{=\!=} \frac{1}{2}\, \dot{\partial}_i\, \dot{\partial}_j\, F^2(x, \dot{x}) \quad (\dot{\partial}_i \equiv \partial/\partial \dot{x}^i)\,. \tag{1.1}$$

Let us consider further a tensor field $T^i_j(x, \dot{x})$ depending both upon positional and directional arguments. The projective covariant derivative [7] of $T^i_j(x, \dot{x})$ with respect to x^k is given by

$$T^i_{j((k))} = \partial_k T^i_j - \dot{\partial}_m T^i_j \Pi^m_{\gamma k}\, \dot{x}^\gamma + T^h_j \Pi^i_{hk} - T^i_h \Pi^h_{jk}\,, \tag{1.2}$$

where

$$\Pi^i_{hk}(x, \dot{x}) \stackrel{\text{def.}}{=\!=} \left\{ G^i_{hk} - \frac{1}{(n+1)} (2\, \delta^i_{(h} G^\gamma_{k)\gamma} + \dot{x}^i G^\gamma_{\gamma kh}) \right\} \tag{1.3}$$

are projective connection coefficients and satisfy the following equations :

$$\text{a) } \Pi^i_{hjk} = \dot{\partial}_h \Pi^i_{jk}, \quad \text{b) } \Pi^i_{hjk}\, \dot{x}^h = 0\,, \quad \text{c) } \Pi^i_{hjk} = \Pi^i_{jkh} = \Pi^i_{khj}\,. \tag{1.4}$$

The commutation formula involving the projective covariant derivative is given by [7] :

$$2T^i_{j[((h))((k))]} = -\,\dot{\partial}_\gamma T^i_j Q^\gamma_{hk} + T^s_j Q^i_{shk} - T^i_s Q^s_{jhk}\,, \tag{1.5}$$

where

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1) The numbers in brackets refer to the references given at the end of the paper.

$$Q^i_{hjk}(x,\dot{x}) \overset{\text{def.}}{=} 2\{\partial_{[k}\Pi^i_{j]h} - \Pi^i_{\gamma h[j}\Pi^\gamma_{k]} + \Pi^\gamma_{h[j}\Pi^i_{k]\gamma}\} \tag{1.6}$$

are called the curvature tensors and satisfy the following identities [7] in an affinely connected space :

$$Q^i_{hjk((s))} + Q^i_{hks((j))} + Q^i_{hsj((k))} = 0\,, \tag{1.7}$$

$$Q^i_{hjk} + Q^i_{jkh} + Q^i_{khj} = 0\,, \tag{1.8}$$

$$\text{a)}\quad Q^i_{hjk} = -Q^i_{hkj}\,,\qquad \text{b)}\quad Q^i_{hjk}\,\dot{x}^h = Q^i_{jk} \text{ and c)}\quad Q_{hj} = Q^i_{hji}\,. \tag{1.9}$$

Let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\,dt\,, \tag{1.10}$$

where $v^i(x)$ is any vector field and dt is an infinitesimal point constant. The Lie-derivatives of $T^i_j(x,\dot{x})$ and $\Pi^i_{jk}(x,\dot{x})$ are given by

$$\pounds v T^i_j = T^i_{j((h))}v^h - T^h_j v^i_{((h))} + T^i_h v^h_{((j))} + \dot{\partial}_h T^i_j v^h_{((s))}\dot{x}^s \tag{1.11}$$

and

$$\pounds v\Pi^i_{jk} = v^i_{((j))((k))} - Q^i_{jkh}v^h + \Pi^i_{sjk}v^s_{((\gamma))}\dot{x}^\gamma \tag{1.12}$$

respectively.

The commutation formula involving the projective covariant derivative and Lie-derivative for $\Pi^i_{jk}(x,\dot{x})$ is given by

$$(\pounds v\Pi^i_{jh})_{((k))} - (\pounds v\Pi^i_{kh})_{((j))} = \pounds vQ^i_{hjk} + 2\dot{x}^s\Pi^i_{\gamma h[j}\pounds v\Pi^\gamma_{k]s}\,. \tag{1.13}$$

If the curvature tensor $Q^i_{hjk}(x,\dot{x})$ of the space satisfies the relation

$$Q^i_{hjk((s))} = \mu_s Q^i_{hjk}\,, \tag{1.14}$$

where $\mu_s(x)$ is a non-zero covariant vector, such type of form is called a special projective recurrent one with respect to μ_s. For brevity, we shall denote such type of Finsler space by S-PRF$_n$-space throughout this paper.

With respect to the infinitesimal point transformation (1.10), we have the following well known theorem :

Theorem (1.1). In order that (1.10) be an infinitesimal special projective affine motion in an F_n, it is necessary and sufficient that Lie-derivative of Π^i_{jk} with respect to (1.10) vanishes :

$$\pounds v\Pi^i_{jk} = v^i_{((j))((k))} - Q^i_{jkh}v^h + \Pi^i_{sjk}v^s_{((\gamma))}\dot{x}^\gamma = 0\,. \tag{1.15}$$

The above equation, of course, can be applied to the space F_n under consideration. The integrability condition of the equation (1.15) is given by

$$(\pounds\, v\Pi^i_{jk})_{((l))} - (\pounds\, v\Pi^i_{jl})_{((k))} = 0\,, \tag{1.16}$$

which in view of the equations (1.11) and (1.13) reduces to

$$\pounds v Q^i_{hjk} = Q^i_{hjk((s))} v^s - Q^s_{hjk} v^i_{((s))} + Q^i_{sjk} v^s_{((h))} + \\ + Q^i_{hsk} v^s_{((j))} + Q^i_{hjs} v^s_{((k))} + \dot\partial_s Q^i_{hjk} v^s_{((\gamma))} \dot x^\gamma = 0\,. \tag{1.17}$$

Introducing the recurrency definition (1.14) into the above equation, we have

$$\pounds v Q^i_{hjk} = \mu_s v^s Q^i_{hjk} - Q^s_{hjk} v^i_{((s))} + Q^i_{sjk} v^s_{((h))} + \\ + Q^i_{hsk} v^s_{((j))} + Q^i_{hjs} v^s_{((k))} + \dot\partial_s Q^i_{hjk} v^s_{((\gamma))} \dot x^\gamma = 0\,. \tag{1.18}$$

In the following, we shall study on the possibility of existence of such a motion. Let us consider the definition (1.14) as a partial differential equation with respect to Q^i_{hjk}, then we can take here a condition

$$0 = (Q^i_{hjk((s))} - \mu_s Q^i_{hjk})_{((m))} - (Q^i_{hjk((m))} - \mu_m Q^i_{hjk})_{((s))}\,. \tag{1.19}$$

In view of the definition (1.14) and the commutation formula (1.5), this reduces to

$$-(\mu_{s((m))} - \mu_{m((s))}) Q^i_{hjk} - \dot\partial_\gamma Q^i_{hjk} Q^\gamma_{lsm} \dot x^l + Q^\gamma_{hjk} Q^i_{\gamma sm} - \\ - Q^i_{\gamma jk} Q^\gamma_{hsm} - Q^i_{h\gamma k} Q^\gamma_{jsm} - Q^i_{hj\gamma} Q^\gamma_{ksm} = 0\,. \tag{1.20}$$

If μ_s denotes a gradient vector given by $\dfrac{1}{\mu}\mu_{((s))}$ $(\mu = \mu(x))$, (1.20) becomes

$$-\dot\partial_\gamma Q^i_{hjk} Q^\gamma_{lsm} \dot x^l + Q^\gamma_{hjk} Q^i_{\gamma sm} - Q^i_{\gamma jk} Q^\gamma_{hsm} - Q^i_{h\gamma k} Q^\gamma_{jsm} - \\ - Q^i_{hj\gamma} Q^\gamma_{ksm} = 0\,. \tag{1.21}$$

The equations (1.18) and (1.21) show that if we can take

$$v^i_{((j))} = Q^i_{jkh} t^{kh}\,, \tag{1.22}$$

where t^{kh} is any suitable non-symmetric tensor, then

$$\mu_s v^s = 0\,. \tag{1.23}$$

This is re-written also as

$$\pounds v \mu_s(x) = 0\,. \tag{1.24}$$

2. Concurrent Field. Under a concurrent field we mean a field characterized by

$$v^i_{((j))} = b\delta^i_j \qquad (b = \text{non-zero constant}). \tag{2.1}$$

If there is a motion of the type

$$\bar{x}^i = x^i + v^i(x)\,dt\,, \qquad v^i_{((j))} = b\,\delta^i_j \tag{2.2}$$

we have

$$v^i_{((j))((k))} - v^i_{((k))((j))} = 0 \tag{2.3}$$

or

$$Q^i_{hjk}\,v^h = 0\,, \tag{2.4}$$

where we have used the equation (1.15).

Differentiating (2.4) projective covariantly with respect to x^s and noting the equations (1.14), (2.2) and (2.4) itself, we get

$$bQ^i_{hjk} = 0\,. \tag{2.5}$$

Hence the space becomes a flat one. Therefore, there does not exist a special projective affine motion (2.2) in an S-PRF$_n$-space.

3. **Special Concircular Field.** Under a special concircular field, we study a vector field defined by

$$v^i_{((j))} = \psi(x)\,\delta^i_j\,, \tag{3.1}$$

where $\psi(x)$ is an arbitrary non-zero scalar function. Let us consider a system of motion of the following form :

$$\bar{x}^i = x^i + v^i(x)\,dt\,, \quad v^i_{((j))} = \psi(x)\,\delta^i_j\,. \tag{3.2}$$

If it will be the case, then with the help of the last relation, we can see

$$v^i_{((j))((k))} - v^i_{((k))((j))} = \psi_{((k))}\,\delta^i_j - \psi_{((j))}\,\delta^i_k \tag{3.3}$$

which in view of the equation (1.5) reduces to

$$v^h\,Q^i_{hjk} = \psi_{((k))}\,\delta^i_j - \psi_{((j))}\,\delta^i_k\,. \tag{3.4}$$

Differentiating this equation covariantly with respect to x^m and noting the equations (1.14), (3.2) and (3.4) itself, we obtain

$$\psi Q^i_{mjk} + \mu_m(\psi_{((k))}\,\delta^i_j - \psi_{((j))}\,\delta^i_k) = \psi_{((k))((m))}\,\delta^i_j - \psi_{((j))((m))}\,\delta^i_k\,. \tag{3.5}$$

The above equation can also be written as

$$\psi Q^i_{mjk} = \delta^i_j\,(\psi_{((k))((m))} - \mu_m\,\psi_{((k))}) - (\psi_{((j))((m))} - \mu_m\,\psi_{((j))})\,\delta^i_k. \tag{3.6}$$

Hence, if we take $\psi_{((k))((m))} = \mu_m \psi_{((k))}$ the last equation reduces to

$$Q^i_{mjk} = 0\,. \tag{3.7}$$

Hence, in order to avoid a case where the space be reduced to a flat P_n, we have to assume $\psi_{((k))((m))} \neq \mu_m \psi_{((k))}$. This is to say, the gradient vector $\psi_{((k))}$ is not a recurrent with respect to μ_m. Under such an assumption, we can consider (3.2) as usual. However, unfortunately, if $v^i_{((j))} = \psi(x)\,\delta_j^i$, $\psi(x)$ vanishes identically [3] [2].

Consequently, for a $\psi(x) \neq 0$, there does not exist a special projective affine motion of the form (3.2) in S-PRF$_n$.

4. Recurrent Field. If a vector $v^i(x)$ satisfies the condition

$$v^i_{((j))} = \psi_j(x)\, v^i\,, \tag{4.1}$$

where ψ_j denotes an arbitrary covariant vector, then the vector field spanned by v^i satisfying (4.1) is called a recurrent field. In this section, we shall study the possibility of special projective affine motion of the form

$$\bar{x}^i = x^i + v^i(x)\, dt,\ v^i_{((j))} = \psi_j(x)\, v^i\,. \tag{4.2}$$

If there exists such a motion in S-PRF$_n$ first of all v^i has to satisfy (1.15) ,hence introducing (4.1) into (1.15), we get

$$Q^i{}_{jkh}\, v^h = (\psi_{j\,((k))} + \psi_j\,\psi_k)\, v^i\,, \tag{4.3}$$

where we have used (4.1) in the process of calculation. Since $Q^i{}_{hjk}$ is anti-symmetric in j and k, multiplying the last equation by v^k we obtain

$$\psi_{j\,((k))}\, v^i\, v^k + \psi_j\,\psi_k\, v^i\, v^k = 0 \tag{4.4a}$$

from which it follows that

$$\psi_{j\,((k))}\, v^k + \psi_j\,\psi_k\, v^k = 0. \tag{4.4b}$$

In view of the equations (1.14) and (4.1), differentiating (4.3) projectively with respect to x^m, we get

$$\begin{aligned}\psi_{j\,((k))\,((m))}\, v^i + \psi_{j\,((k))}\,\psi_m\, v^i + \psi_{j\,((m))}\,\psi_k\, v^i + \\ + \psi_j\,\psi_{k\,((m))}\, v^i + \psi_j\,\psi_k\,\psi_m\, v^i = (\mu_m + \psi_m)\, Q^i{}_{jkh}\, v^h\,.\end{aligned} \tag{4.5}$$

Contracting the above equation with respect to the indices i and k we obtain

$$\begin{aligned}\psi_{j\,((k))\,((m))}\, v^k + \psi_{j\,((k))}\,\psi_m\, v^k + \psi_{j\,((m))}\,\psi_k\, v^k + \\ + \psi_j\,\psi_{k\,((m))}\, v^k + \psi_j\,\psi_k\,\psi_m\, v^k = -\,Q_{jh}\,(\mu_m + \psi_m)\, v^h\,.\end{aligned} \tag{4.6}$$

Again differentiating (4.4b) covariantly, with respect to x^m and using the equation (4.1), we get

$$\begin{aligned}\psi_{j\,((k))\,((m))}\, v^k + \psi_{j\,((k))}\,\psi_m\, v^k + \psi_{j\,((m))}\,\psi_k\, v^k + \\ + \psi_j\,\psi_{k\,((m))}\, v^k + \psi_j\,\psi_k\,\psi_m\, v^k = 0\,.\end{aligned} \tag{4.7}$$

Comparing the last equation with (4.6), we can find a remarkable result

[2] See theorem 3 and its generalization in [3].

$$(\mu_m + \psi_m) Q_{jh} v^h = 0, \tag{4.8}$$

from which we can obtain

$$\mu_m = -\psi_m \quad \text{or} \quad Q_{jh} v^h = 0\ ^{3)}. \tag{4.9}$$

For the first case, we have

$$\bar{x}^i = x^i + v^i(x)\,dt\,, \quad v^i_{((j))} = -\mu_j v^i\,. \tag{4.10}$$

In the following lines, we shall seek for a necessary and sufficient condition for the existence of special projective affine motion (4.10). In order that (4.10) construct a special projective affine motion, it is necessary and sufficient that (4.10) satisfy (1.18). In order to find an essential condition for our purpose, let us introduce (4.10) into the left hand side of (1.18). Then, we have

$$£vQ^i_{hjk}(x, \dot{x}) = \mu_s v^s Q^i_{hjk} + \mu_s Q^s_{hjk} v^i - \mu_h Q^i_{sjk} v^s - \mu_j Q^i_{hsk} v^s - \mu_k Q^i_{hjk} v^s$$

$$£vQ^i_{hjk} = v^s(\mu_s Q^i_{hjk} - \mu_j Q^i_{hsk} - \mu_k Q^i_{hjs}) + \mu_s Q^s_{hjk} v^i - \mu_h Q^i_{sjk} v^s \tag{4.11}$$

$$= v^s(Q^i_{hjk((s))} + Q_{hks((j))} + Q^i_{hsj((k))}) + Q^s_{hjk} v^i{}_{((s))} - Q^i_{sjk} v^s{}_{((h))}\,,$$

where we have used the equations (1.14) and (4.10). In an affinely connection space the identity (1.7) takes the form

$$\mu_s Q^i_{hjk} + \mu_j Q^i_{hks} + Q^i_{hsj}\mu_k = 0\,. \tag{4.12}$$

Therefore, in view of the above relation the equation (4.11) reduces to

$$£vQ^i_{hjk} = -v^i{}_{((s))} Q^s_{hjk} + v^s{}_{((h))} Q^i_{sjk}\,. \tag{4.13}$$

On the other hand, we have generally

$$v^i_{((s))} Q^s_{hjk} - v^s_{((h))} Q^i_{sjk} = v^i_{((h))((j))((k))} - v^i_{((h))((k))((j))}. \tag{4.14a}$$

Hence, in order to get $£v\, Q^i_{hjk} = 0$, in the present case it is necessary and sufficient that

$$v^i_{((h))((j))((k))} - v^i{}_{((h))((k))((j))} = 0 \tag{4.14b}$$

and in this case only (4.10) becomes a special projective affine motion of the space. The relation (4.14b) is an integrability condition of the equation

$$(\alpha(x)\, v^i{}_{((h))})_{((j))} = 0\,, \tag{4.15a}$$

where $\alpha(x)$ is a non-zero arbitrary function of x'^s only. Consequently, in order to get the special projective affine motion (4.10), it is necessary and sufficient that (4.15a) be assumed. Introducing the form of (4.10) into (4.15a), we obtain

[3] We shall consider afterward the case where $Q_{jh} v^h = 0$. See §7.

$$v^i(-\alpha_{((j))}\mu_h - \alpha\mu_{h((j))} + \alpha\mu_h\mu_j) = 0\,. \tag{4.16}$$

So neglecting non-zero v^i, we have

$$\mu_{h((j))} = \mu_h\mu_j - \alpha_j\mu_h \qquad (\alpha_j \equiv \alpha_{((j))} \mid \alpha)\,. \tag{4.17}$$

Therefore, we can now consider here a motion

$$\bar{x}^i = x^i + v^i(x)\,dt,\ v^i_{((j))} = -\mu_j v^i,\ \mu_{h((j))} = \mu_h p_j - \alpha_j\mu_h\,. \tag{4.18}$$

Conversely, if we have (4.18), from this form, we can obtain the original form (4.15a).

If we are able to consider (4.14b) excepting for a scalar proportionality $\alpha(x)$, that is, $\alpha_j = 0$ in place of (4.18), we get

$$\bar{x}^i = x^i + v^i(x)\,dt \quad , \quad v^i_{((j))} + \mu_j v^i = 0\,,\ \mu_{h((j))} = \mu_h\mu_j\,. \tag{4.19}$$

The last of the above equations shows that μ_j be a gradient :

$$\mu_j = \frac{1}{\mu}\mu_{((j))} \quad , \quad (\mu = \mu(x^h))\,. \tag{4.20}$$

In this case, we have $(v^\gamma\mu_\gamma)_{((h))} = v^\gamma(\mu_{\gamma((h))} - \mu_\gamma\mu_h) = 0$, i.e. $v^\gamma\mu_\gamma = \text{const.}$ Taking this constant to be zero, we obtain $v^\gamma\mu_\gamma = 0$ or $\pounds v\mu(x) = 0$. Thus ,we have :

Theorem (4.1). An S-PRF$_n$-space is able to admit a special projective affine motion of recurrent form :

$$\bar{x}^i = x^i + v^i(x)\,dt\,, \quad v^i_{((j))} + \mu_j v^i = 0$$

with an additional equation (4.17) being assumed to be integrable.

Corollary (4.1). The S-PRF$_n$-space is able to have a special projective affine motion of contra form

$$\bar{x}^i = x^i + v^i(x)\,dt \quad , \quad v^i_{((h))} + \mu_h v^i = 0$$ [4)]

with the conditions

$$\text{a)}\quad \mu_{h((s))} = \mu_h\mu_s \quad , \quad \text{b)}\quad \pounds v\mu(x) = 0 \tag{4.21}$$

where $\mu_j = \mu_{((j))} \mid \mu$.

Such a motion has been introduced under the solvability of the characteristic equation (4.17). However, (4.17) has actually a special solution $\mu_h = 0$. Hence the following existence theorem of a contramotion holds good.

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4) This is re-written as $(\mu v^i)_{((j))} = 0$, so v^i spans a contra field (except for a scalar proportionality). In case of $\mu = \text{Const.}$, we shall call such a field a contra field in the strict sense. Under a contra-field, we understood a field of parallel contravariant vectors.

Theorem (4.2). The space S-PRF$_n$ is able to have naturally a special projective affine motion of contra form in the strict sense $\bar{x}^i = x^i + v^i(x)\,dt$, $v^i_{((j))} = 0$.

Since $\mu_j = 0$ means that S-PRF$_n$ be a special projective symmetric Finsler space [6], we have :

Theorem (4.3). In order that a S-PRF$_n$-space admitting a special projective affine motion of recurrent form of the type (4.18) be a special projective symmetric space, it is necessary and sufficient that the motion be taken to be a contra form in the strict sense.

5. Some Essential Conditions. Let us discuss the characteristic differential equation (4.17) of the defining vector μ_h for the recurrent affine motion. The integrability condition of this equation is given by

$$(\mu_{h((j))} - \mu_h \mu_j + \alpha_j \mu_h)_{((s))} - (\mu_{h((s))} - \mu_h \mu_s + \alpha_s \mu_h)_{((j))} = 0 \tag{5.1}$$

which by virtue of the commutation formula (1.5) reduces to

$$\mu_\gamma Q^\gamma{}_{hjs} = (\mu_{j((s))} - \mu_{s((j))})\,\mu_h \tag{5.2}$$

i.e.

$$\mu_\gamma Q^\gamma{}_{hjs} = (-\alpha_s \mu_j + \alpha_j \mu_s)\,\mu_h\,. \tag{5.3}$$

If we take the arbitrary α_s as a gradient vector being equal to zero or μ_s then the last two equations reduce to

$$\mu_\gamma Q^\gamma{}_{hjs} = 0\,. \tag{5.4}$$

But in such case the integrability conditon (4.17) becomes

$$\mu_{h((j))} = \mu_h \mu_j \quad \text{or} \quad \mu_{h((j))} = 0 \tag{5.5}$$

and (5.4) holds identically.

The equation (4.18) is equivalent to the system of (4.10) and (4.15a), i.e. to that of (4.10) and

$$\alpha_j v^i{}_{((h))} + v^i{}_{((h))((j))} = 0\,. \tag{4.15b}$$

Introducing the value of $v^i{}_{((h))((j))}$ from (1.15) into the above equation, we get

$$\alpha_j v^i{}_{((h))} = -Q^i{}_{hjs} v^s + \Pi^i{}_{shj} v^s{}_{((\gamma))} \dot{x}^\gamma\,. \tag{5.6}$$

In view of the equation (4.10), the last relation can be written like

$$\mu_h \alpha_j v^i = Q^i{}_{hjs} v^s\,. \tag{5.7}$$

Transvecting this equation by v^j and noting the relation $Q^i{}_{hjs} v^j v^s = 0$ we obtain

$$\alpha_j v^j = 0 \qquad \text{or} \qquad \pounds v\, \alpha(x) = 0 \,. \tag{5.8}$$

Since μ_h and v^i are non-zero, the above equation shows that $\alpha(x)$ is a Lie-invariant one.

Contracting the equation (5.7) with respect to the indices i and j and using (5.8), we get

$$0 = \mu_h \alpha_j v^j = Q^i_{his} v^s = - Q_{hs} v^s \,, \tag{5.9}$$

i.e.

$$Q_{hs} v^s = 0 \,. \tag{5.10}$$

In an affinely connected space the Bianchi identity (1.7) reduces to

$$Q^i_{hjk((s))} + Q^i_{hks((j))} + Q^i_{hsi((k))} = 0 \,. \tag{5.11}$$

Contracting the above equation with respect to the indices i and k and using the relation (1.9), we get

$$Q_{hj((s))} - Q_{hs((j))} = - Q^i_{hjs((i))} \tag{5.12}$$

which in view of the definition (1.14) reduces to

$$\mu_i Q^i_{hjs} = - \mu_s Q_{hj} + \mu_j Q_{hs} \,. \tag{5.13}$$

Comparing this equation with (5.3), we obtain

$$\mu_s (Q_{hj} - \alpha_j \mu_h) = \mu_j (Q_{hs} - \mu_h \alpha_s) \,. \tag{5.14}$$

Multiplying this equation by v^s and summing over s, we get

$$\mu_s v^s (Q_{hj} - \mu_h \alpha_j) = 0 \,, \tag{5.15}$$

where we have used $\pounds v\alpha(x) = 0$ and (5.10). Hence if S-PRF$_n$-space admits the special projective affine motion of the recurrent form (4.18), we have

$$Q_{hj} = \mu_h \alpha_j \qquad \text{or} \qquad \mu_s v^s = 0 \,. \tag{5.16}$$

Furthermore, let us study the integrability condition of $v^i_{((j))} + \mu_j v^i = 0$.

From $(v^i_{((j))} + \mu_j v^i)_{((k))} - (v^i_{((k))} + \mu_k v^i)_{((j))} = 0$, we get

$$v^h Q^i_{hjk} = v^i (- \mu_{j((k))} + \mu_{k((j))}) \,, \tag{5.17}$$

where we have used the equations (1.5) and (4.18). The last relation can also be written like

$$v^h Q^i_{hjk} = v^i (- \alpha_k \mu_j + \alpha_j \mu_k) \,. \tag{5.18}$$

In view of the equation (1.9a) multiplying the identity (1.8) by v^h, we get

$$v^h Q^i_{hjk} = - Q^i_{jkh} v^h + Q^i_{kjh} v^h . \qquad (5.19)$$

Introducing (5.7) into the left hand side of this equation, we obtain (5.18), i.e. the integrability condition (5.18) holds identically.

6. **Discussion on the vector.** We have assumed the existence of a gradient vector α_j derived from an arbitrary function $\alpha(x)$ satisfying (4.15a) and in the course of discussion of the integrability conditions of (4.17), we have taken up the following two cases :

$$\text{a) } \alpha_j = 0 , \qquad \text{b) } \alpha_j = \mu_j , \qquad (6.1)$$

but we have still remained to prove whether these are possible or not.

Differentiating (1.15) projective covariantly with respect to x^s, and using the equations (4.18), we get

$$v^i_{((j))((k))((s))} = - \mu_s Q^i_{jkh} v^h + \mu_s Q^i_{jkh} v^h = 0 . \qquad (6.2)$$

Again, differentiating projectively (4.15b) with respect to x^s and the equation (4.17b) itself, we obtain

$$\mu_h v^i (\alpha_{j((s))} - \alpha_j \alpha_s) = 0 . \qquad (6.3)$$

Consequently if the S-PRF$_n$-space admits a special projective affine motion of the form (4.18) the above equation reduces to

$$\alpha_{j((s))} - \alpha_j \alpha_s = 0 \qquad (6.4)$$

for a non-zero v^i and μ_h.

This is characteristic equation of $\alpha(x)$ introduced by (4.15a). The integrability condition of this equation is given by

$$\alpha_m Q^m{}_{kjs} = 0 . \qquad (6.5)$$

Thus, we have to assume (6.4) and (6.5) in our discussion of special projective affine motion. At this moment we can now discuss the case (6.1a), (6.1b) and (5.16). At first (6.4) has a solution $\alpha_j = 0$, and therefore, the case (6.1a) is always possible. Secondly, if $\alpha_j = \mu_j$, the equation (6.4) takes the form

$$\mu_{j((s))} = \mu_j \mu_s . \qquad (6.6)$$

On the other hand, in this case, we get from (4.17)

$$\mu_{j((s))} = 0 . \qquad (6.7)$$

Hence from (6.6), we obtain $\mu_j = 0$. Hence a non-special projective symmetric S-PRF$_n$ (i.e. $Q^i_{hjk((s))} = 0$) can not admit the case (6.1b) and when it is the case S-PRF$_n$ must be a special projective symmetric space.

Thus, we can see that the case of $\alpha_j = 0$ is able to admit a contra motion in a general S-PRF$_n$-space, but the case $\alpha_j = \mu_j$ is unable to consider in a general S-PRF$_n$. Accordingly, if we treat only a general non-symmetric space, we have to regard α_j as a vector being not equal to μ_j.

We shall now study the case (5.16). At first we shall discuss the latter :

$$\mu_s v^s = 0 \qquad (\mu_s \neq 0)\,. \tag{6.8}$$

Since we have always $\pounds v \alpha(x) = 0$ or $\alpha_s v^s = 0$, we may consider a special case such that μ_j is proportional to α_j for a non-zero and non-constant $\varepsilon = \varepsilon(x)$[5], i.e.

$$\alpha_i = \varepsilon \mu_j\,. \tag{6.9}$$

In such a case, (4.17) takes the following form :

$$\mu_{h((j))} = (1 - \varepsilon)\, \mu_h \mu_j\,. \tag{6.10}$$

With the help of the last equation, we can obtain

$$\mu_{h((j))} - \mu_{j((h))} = 0, \tag{6.11}$$

that is, μ_h becomes a gradient vector. Then, let us determine the concrete form of μ_h. In this case the integrability condition of (4.17) becomes

$$\mu_\gamma Q^\gamma{}_{hjk} = 0\,. \tag{6.12}$$

On the other hand the deformed equation of (4.17) showed by (6.10) gives us its integrability condition of the form

$$\mu_s Q^s{}_{hjk} = (\mu_j \varepsilon_{((k))} - \mu_k \varepsilon_{((j))})\, \mu_h\,. \tag{6.13}$$

Consequently μ_j must satisfy

$$\mu_i = \lambda \varepsilon_{((k))} \qquad (\lambda = \lambda(x) : \text{suitable function})\,. \tag{6.14}$$

Substituting $\alpha = \varepsilon\mu_j$ into the equation (6.4) and noting (4.17), we obtain

$$\varepsilon_{((m))} = (2\varepsilon - 1)\ \varepsilon\mu_m\,. \tag{6.15}$$

Hence, we can see $\lambda = \dfrac{1}{\varepsilon(2\varepsilon - 1)}$, consequently

$$\mu_s = \frac{\varepsilon_{((s))}}{\varepsilon(2\varepsilon - 1)} = -\frac{1}{\varepsilon}\varepsilon_{((s))} + \frac{2}{2\varepsilon - 1}\varepsilon_{((s))}\,, \tag{6.16}$$

that is μ_j denotes certainly a gradient. Furthermore, being

[5]) When $\varepsilon =$ Const., from (4.17) and (6.4), we get $\varepsilon = 1/2$ or $\mu_j = 0$. These yield trivial cases, so we except the case $\varepsilon =$ Const.

$$\mu_s = \frac{1}{\varepsilon} \cdot \frac{1}{\alpha} \cdot \alpha_{((s))}. \tag{6.17}$$

Equating the two values of μ_s from the equations (6.16) and (6.17), we get

$$\frac{1}{2\varepsilon - 1} \varepsilon_{((s))} = \frac{1}{\alpha} \alpha_{((s))} \,. \tag{6.18}$$

We can find especially at last $\varepsilon = \frac{1}{2}(1 + \alpha^2)$.

7. The Condition $Q_{hj} v^j = 0$ and Integrability of $Q_{hj} = \mu_h \alpha_j$. In §4 in order to find the form of the motion of the form (4.2) we have derived the conditions (4.9). Here the latter condition has been excepted from our study. However, as we can see from (5.10), from the former condition the latter condition always follows. Hence the first condition is a special one of the second condition.

We shall now try to study only the form

$$v^s Q_{hs} = 0 \,. \tag{7.1}$$

In this case, we shall show that we can associate naturally this exceptional case itself with our present theory. In fact, if (7.1) will be the case, differentiating it projective covariantly with respect to x^m, we get

$$(\mu_m + \psi_m) Q_{hs} v^s = 0 \,, \tag{7.2}$$

where we have used (1.14) and (4.18).

From (7.2), we can conclude that the quantity inside the bracket can be taken quite arbitrary. So we can put

$$\mu_m + \psi_m = 0 \qquad \text{or} \qquad \psi_m = - \mu_m \,. \tag{7.3}$$

In view of the above result the recurrent condition (4.1) becomes

$$v^i_{((j))} + \mu_j v^i = 0 \,. \tag{7.4}$$

In this way, we can associate the case (7.1) with our stand point. Now, the integrability condition of

$$Q_{hj} = \mu_h \alpha_j \qquad \text{or} \qquad \alpha Q_{hj} = \mu_h \alpha_{((j))} \tag{7.5}$$

will be calculated and proved with case. That is, at first we make

$$(\mu_{h((s))} \alpha_{((j))} + \mu_h \alpha_{((j))((s))} - Q_{hj} \alpha_{((s))} - \alpha \mu_s Q_{hj}) = 0 \,. \tag{7.6}$$

Introducing the equation (6.4) into the last relation, we get

$$\mu_{h((s))} \alpha_{((j))} + \mu_h (\alpha_{((j))((s))} + \alpha \alpha_s \alpha_j) - \mu_h \alpha_s \alpha_{((j))} - \alpha \mu_s \mu_h \alpha_j = 0 \,. \tag{7.7}$$

Interchanging the indices j and s in the above equation, we get a similar equation. Substracting this equation from (7.7), we get the integrability condition of the form :

$$\alpha_{((j))}(\mu_{h((s))} - \mu_h \mu_s + \alpha_s \mu_h) - (\mu_{h((j))} - \mu_h \mu_j + \alpha_j \mu_h)\,\alpha_{((s))} = 0\,. \qquad (7.8)$$

But in order to get the present motion, we have assumed (4.17), hence above condition holds identically. That is, (7.5) is completely integrable.

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ÖZET

Bu çalışmada bir PRF_n uzayında özel bir infinitezimal projektif afin hareketin varlığı araştırılmaktadır.