PROJECTIVE AFFINE MOTION IN A PRF_n-SPACE, III

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In this paper the existence of an infinitesimal affine motion of concircular form

$$\bar{x}^i = x^i + v^i(x)\,dt,\; v^i{}_{(j)} = \beta\delta^i{}_j + \phi_j(x)\,v^i$$

is studied, where $\beta(x)$ is any function and $\phi_j(x)$ denotes a gradient vector defined by

$$\phi_j(x) = \phi_{(j)} = \partial_j\,\phi\,.$$

1. Introduction. Let us consider an n-dimensional affinely connected Finsler space F_n [1] [1)] having a fundamental positively homogeneous metric function $F(x,\dot{x})$ which satisfies all the requisite conditions imposed upon it. The fundamental metric tensor of the space is given by

$$g_{ij}(x,\dot{x}) \overset{\text{def.}}{=} \frac{1}{2}\,\dot{\partial}_i\,\dot{\partial}_j\,F^2(x,\dot{x}) \tag{1.1}$$

where $\dot{\partial}_i \equiv \partial/\partial\dot{x}^i$.

Let us further consider a tensor field $T_j^i(x,\dot{x})$ which depends both upon positional and directional arguments. The covariant derivative of $T_j^i(x,\dot{x})$ in the sense of Berwald with respect to x^k is given by

$$T^i_{j(k)} = \partial_k\,T_j^i - \dot{\partial}_m\,T_j^i\,G_k^m + T_j^s\,G^i_{sk} - T_s^i\,G^s_{jk}\,, \tag{1.2}$$

where $G^i_{jk}(x,\dot{x})$ are called Berwald's connection coefficients. The commutation formula involving the above covariant derivative is given by

$$2T^i_{j[(h)(k)]} = -\,\dot{\partial}_\gamma\,T_j^i\,H^\gamma_{hk} + T_j^s\,H^i_{shk} - T_s^i\,H^s_{jhk}\,, \tag{1.3}$$

where

$$H^i_{hjk}(x,\dot{x}) \overset{\text{def.}}{=} 2\,\{\partial_{[k}G^i_{j]h} - G^i_{\gamma h[j}G^\gamma_{k]} + G_h^\gamma{}_{[j}G^i_{k]\gamma}\}^{2)} \tag{1.4}$$

is called Berwald's curvature tensor and satisfies the following relations [1]:

1) Numbers in brackets refer to the references at the end of the paper.

2) $2A_{[hk]} = A_{hk} - A_{kh}$

$$H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0\,, \tag{1.5}$$

$$H^i_{hjk(s)} + H^i_{hks(j)} + H^i_{hsj(k)} = G^i_{\gamma hj}\, H^\gamma_{sk} + G^i_{\gamma hk}\, H^\gamma_{js} + G^i_{\gamma hs}\, H^\gamma_{kj}\,, \tag{1.6}$$

$$H^i_{hjk} = -\, H^i_{hkj}\,. \tag{1.7}$$

Now, let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i\,(x)\, dt\,,$$

where $v^i\,(x)$ is any vector field and dt is an infinitesimal point constant. In view of the above point transformation and Berwald's covariant derivative, Lie-derivatives of $T^i_j\,(x,\dot{x})$ and $G^i_{jk}\,(x,\dot{x})$ are given by

$$\pounds v\; T_j^i = T^i_{j(h)}\, v^h + T_h^i\, v^h_{(j)} - T_j^h\, v^i_{(h)} + \dot{\partial}_h\, T_j^i\, v^h_{(\gamma)}\, \dot{x}^\gamma \tag{1.9}$$

and

$$\pounds v\; G^i_{jk} = v^i_{(j)\,(k)} + H^i_{jkh}\, v^h + G^i_{hjk}\, v^h_{(\gamma)}\, \dot{x}^\gamma \tag{1.10}$$

respectively.

If the Berwald's curvature tensor $H^i_{hjk}\,(x,\dot{x})$ satisfles the relation

$$H^i_{hjk\,(s)} = \lambda_s\, H^i_{hjk}\,, \tag{1.11}$$

where $\lambda_s\,(x)$ means a non-zero covariant vector then the space F_n is called a projective recurrent Finsler space or PRF_n-space and $\lambda_s\,(x)$ is called a projective recurrent vector.

The present author has studied infinitesimal affine motions of many kinds in such an PRF_n-space and obtained the following results [6] :

An PRF_n-space is able to admit an infinitesimal affine motion (1.8) characterized by

$$\text{a)}\quad v^i_{\,(j)} = 0 \qquad \text{b)}\quad v^i_{(j)} = \phi_j(x)\, v^i \tag{1.12}$$

that is, a contraform and a recurrent form respectively, but PRF_n-space is unable to have a motion of concurrent form and special concircular form respectively [5] , [6] :

$$\begin{aligned} &\text{a)}\quad v^i_{(j)} = \rho\, \delta_j^i \quad (\rho = \text{non-zero const.})\,, \\ &\text{b)}\quad v^i_{(j)} = \phi(x)\, \delta_j^i \quad (\phi = \text{non-zero function}). \end{aligned} \tag{1.13}$$

In this manuscript we shall study the existence of an infinitesimal affine motion of concircular form :

$$\bar{x}^i = x^i + v^i(x)\,dt, \quad v^i_{(j)} = \beta\,\delta_j^i + \phi_j(x)\,v^i, \tag{1.14}$$

where $\beta(x)$ is any function and $\phi_j(x)$ denotes a gradient vector defined by

$$\phi_j(x) = \phi_{(j)} = \partial_j \phi\,. \tag{1.15}$$

2. Fundamental Results (1). For an infinitesimal projective affine motion, we have

$$£v\, G^i_{jk} = v^i_{(j)(k)} + H^i_{jkh}\, v^h + G^i_{hjk}\, v^h_{(\gamma)}\, \dot{x}^\gamma = 0\,. \tag{2.1}$$

In an PRF$_n$-space, the integrability condition for the above equation is given by

$$£v\, H^i_{hjk} = \lambda_s\, v^s\, H^i_{hjk} - H^s_{hjk}\, v^i_{(s)} + H^i_{sjk}\, v^s_{(h)} + H^i_{hsk}\, v^s_{(j)} + + H^i_{hjs}\, v^s_{(k)} + \dot{\partial}_s\, H^i_{hjk}\, v^s_{(\gamma)}\, \dot{x}^\gamma\,. \tag{2.2}$$

During the process of study we shall assume here the existence of projective affine motion of concircular form as follows :

$$\bar{x}^i = x^i + v^i(x)\,dt, \qquad v^i_{(j)} = \beta(x)\,\delta^i_j + \phi_j(x)\,v^i, \tag{2.3}$$

where $\beta = \beta(x)$ is a scalar function and $\phi_j(x)$ means a covariant vector depending only upon the positional arguments. From the above definition, we can obtain

$$v^i_{(j)(k)} = \phi_{j(k)}\, v^i + \phi_j\, \phi_k\, v^i + \beta\, \phi_j\, \delta^i_k + \beta_k\, \delta^i_j\,, \tag{2.4}$$

where

$$\beta_k \overset{\text{def.}}{=} \beta_{(k)} = \partial_k \beta \tag{2.5}$$

and we have also used the equation (2.3).

In view of the equation (2.4), the formula (2.1) reduces to

$$H^i_{jkh}\, v^h = \phi_{j(k)}\, v^i + \phi_j\, \phi_k\, v^i + \beta\phi_j\, \delta^i_k + \beta_k\, \delta^i_j\,. \tag{2.6}$$

By virtue of the fact $H^i_{hjk}\, v^j\, v^k = 0$, transvecting the above equation by v^k, we have

$$\phi_{j(k)}\, v^i\, v^k + \phi_j\, \phi_k\, v^i\, v^k + \beta\phi_j\, v^i + \beta_k\, \delta^i_j\, v^k = 0\,. \tag{2.7}$$

Contracting the above equation with respect to the indices i and j, we get

$$\phi_{j(k)}\, v^j\, v^k + \phi_j\, \phi_k\, v^j\, v^k + \beta\phi_j\, v^j + n\beta_k\, v^k = 0\,. \tag{2.8}$$

Now, transvecting the result (2.7) by v^j, we obtain

$$\phi_{j(k)}\, v^i\, v^k\, v^j + \phi_j\, \phi_k\, v^i\, v^k\, v^j + \beta\phi_j\, v^i\, v^j + \beta_k\, v^k\, v^i = 0\,. \tag{2.9}$$

And for a non-vanishing $v^i(x)$, the above equation yields

$$\phi_{j(k)} v^k v^j + \phi_j \phi_k v^k v^j + \beta\phi_j v^j + \beta_k v^k = 0 . \tag{2.10}$$

Now comparing the equations (2.8) and (2.10), we get

$$(n-1)\beta_k v^k = 0 \tag{2.11}$$

or

$$\beta_k v^k = 0 \qquad (n \neq 1) . \tag{2.12}$$

Thus, we have :

Lemma (2.1). If an n-dimensional PRF_n-space admits a projective affine motion of a torse-forming form

$$\bar{x}^i = x^i + v^i(x)\,dt , \quad v^i_{(j)} = \beta(x)\,\delta^i_j + \phi_j(x)\,v^i$$

then the scalar function $\beta(x)$ must be a Lie-invariant-one or $£v\beta = 0$.

In view of the above lemma and the non-vanishing property of the vector $v^i(x)$, the equation (2.7) can be written like

$$\phi_{j(k)} v^k + \phi_j \phi_k v^k + \beta\phi_j = 0 . \tag{2.13}$$

The above formula can also be written in a compact form :

$$(\phi_j v^j)_{(k)} = 0 , \tag{2.14}$$

where we have used (2.3) and the fact that $\phi_{j(k)} = \phi_{k(j)}$. Thus, we have :

Lemma (2.2). If an n-dimensional PRF_n-space admits an projective affine motion of concircular form, $\phi_j v^j$ must be a constant.

Contracting the equation (2.4) with respect to the indices i and k, we have

$$v^h_{(j)(h)} = \phi_{j(h)} v^h + \phi_j(n\beta + \phi_h v^h) + \beta_j . \tag{2.15}$$

Substituting (2.13) into the right hand side of the equation (2.15), we get

$$v^h_{(j)(h)} = (n-1)\beta\phi_j + \beta_j . \tag{2.16}$$

Differentiating the equation (2.6) covariantly with respect to x^s and using the equations (1.11) and (2.3), we obtain

$$(\lambda_s + \phi_s) H^i_{jkh} v^h + \beta H^i_{jks} = \phi_{j(k)(s)} v^i + \phi_{j(k)} v^i_{(s)} + \phi_{j(s)} \phi_k v^i + \\ + \phi_j \phi_{k(s)} v^i + \phi_j \phi_k v^i_{(s)} + \beta_s \phi_j \delta^i_k + \beta\phi_{j(s)} \delta^i_k + \beta_{k(s)} \delta^i_j . \tag{2.17}$$

Contracting the above equation with respect to the indices i and j, we have

$$(\lambda_s + \phi_s) H^i_{ikh} v^h + \beta H^i_{iks} = \phi_{i(k)(s)} v^i + \phi_{i(k)} v^i_{(s)} + \phi_i \phi_{k(s)} v^i + \\ + \phi_i \phi_k v^i_{(s)} + \beta_s \phi_k + \beta\phi_{k(s)} + n\beta_{k(s)} . \tag{2.18}$$

Now, differentiating (2.13) covariantly with respect to x^s and arranging indices appearing in its result suitably, we get

$$\phi_{k(i)(s)} v^i + \phi_{k(i)} v^i_{(s)} + \phi_{k(s)} \phi_i v^i + \phi_k \phi_{i(s)} v^i + \phi_k \phi_i v^i_{(s)} + \\ + \beta_s \phi_k + \beta\phi_{k(s)} = 0 . \qquad (2.19)$$

If we compare these two equations (2.18) and (2.19) with each other and use the property $\phi_{j(k)} = \phi_{k(j)}$, we obtain

$$(\lambda_s + \phi_s) H^i_{ikh} v^h + \beta H^i_{iks} = n\beta_{k(s)} . \qquad (2.20)$$

Using commutativity of $\beta_{k(s)}$ with respect to its indices, we can get an equality

$$(\lambda_s + \phi_s) H^i_{ikh} v^h + \beta H^i_{iks} = (\lambda_k + \phi_k) H^i_{ish} v^h + \beta H^i_{isk} \qquad (2.21)$$

or

$$2\beta H^i_{iks} = (\lambda_k + \phi_k) H^i_{ish} v^h - (\lambda_s + \phi_s) H^i_{ikh} v^h , \qquad (2.22)$$

where we have used (1.7).

Transvecting the equation (2.22) by v^s and noting the fact that $H^i_{hjk} v^j v^k = 0$, we obtain

$$2\beta H^i_{iks} v^s = (\lambda_s + \phi_s) H^i_{ikh} v^s v^h \qquad (2.23)$$

or

$$H^i_{iks} v^s (2\beta + \lambda_h v^h + \phi_h v^h) = 0 . \qquad (2.24)$$

If we assume $H^i_{iks} v^s \neq 0$, the above equation reduces to

$$2\beta + \lambda_h v^h + \phi_h v^h = 0 . \qquad (2.25)$$

In view of the commutation formula (1.3) and the latter part of (2.3), we can construct the relation

$$v^h H^i_{hjk} = \beta(\phi_j \delta^i_k - \phi_k \delta^i_j) + (\beta_k \delta^i_j - \beta_j \delta^i_k) . \qquad (2.26)$$

Contracting the above formula with respect to the indices i and k, we have

$$v^h H^i_{hji} = (n - 1) (\beta\phi_j - \beta_j) . \qquad (2.27)$$

By virtue of the lemma (2.1) transvecting the above equation by v^j, and summing over the index j, we obtain

$$H^i_{hji} v^h v^j = (n - 1) \beta\phi_j v^j . \qquad (2.28)$$

Differentiating (2.28) covariantly with respect to x^m, we have

$$H^i_{hji(m)} v^h v^j + H^i_{hji} v^h_{(m)} v^j + H^i_{hji} v^h v^j_{(m)} = (n-1) \{\beta_m \phi_j v^j + \\ + \beta\phi_{j(m)} v^j + \beta\phi_j v^j_{(m)}\} . \qquad (2.29)$$

In view of the equations (1.11), (2.3), (2.13) and (2.28), the above relation can be written like

$$(n-1)(2\beta\phi_m + \beta\lambda_m - \beta_m)\phi_j v^j = \beta v^j (H^i_{mji} + H^i_{jmi}) . \qquad (2.30)$$

Transvecting the above equation by v^m and using the equation (2.27) and lemma (2.1), we obtain

$$(n-1)(2\beta\phi_m + \beta\lambda_m - \beta_m)\phi_j v^j v^m = 2(n-1)\beta^2 \phi_m v^m \qquad (2.31)$$

or

$$(2\phi_m v^m + \lambda_m v^m + 2\beta)\beta\phi_j v^j = 0 \qquad (n \neq 1) . \qquad (2.32)$$

Substituting the equation (2.25) into the left hand side of the above formula, we get

$$\beta(\phi_j v^j)(\phi_m v^m) = 0 . \qquad (2.33)$$

Hence, we get the two cases being $\beta = 0$ or $\phi_j v^j = 0$.

3. Fundamental Results (2). In the previous section, we have obtained two cases $\beta = 0$ or $\phi_j v^j = 0$, but, unfortunately, the latter condition becomes the former condition. In this section, we shall prove this fact.

In an affinely connected Finsler space, Bianchi identity for Berwald's curvature tensor $H^i_{hjk}(x, \dot{x})$ is given by

$$H^i_{hjk(s)} + H^i_{hks(j)} + H^i_{hsj(k)} = 0 . \qquad (3.1)$$

By virtue of the equations (1.7) and (1.11), the above formula yields

$$\lambda_s H^i_{hjk} - \lambda_j H^i_{hsk} + \lambda_k H^i_{hsj} = 0 . \qquad (3.2)$$

Transvecting the above result by v^k and summing over the index k, we get

$$\lambda_s H^i_{hjk} v^k - \lambda_j H^i_{hsk} v^k + \lambda_k v^k H^i_{hsj} = 0 . \qquad (3.3)$$

By virtue of the equations (2.1) and (3.3) we can get

$$\lambda_s (v^i_{(h)(j)} + G^i_{nhj} v^n_{(\gamma)} \dot{x}^\gamma) - \lambda_j (v^i_{(h)(s)} + G^i_{nhs} v^n_{(\gamma)} \dot{x}^\gamma) + \\ + \lambda_k v^k H^i_{hsj} = 0 . \qquad (3.4)$$

Contracting the above equation with respect to the indices i and h, we have after little simplification

$$\lambda_s v^h_{(h)(j)} - \lambda_j v^h_{(h)(s)} + \lambda_k v^k H^h_{hsj} = 0 . \qquad (3.5)$$

For $\phi_h v^h = 0$, the formula (2.25) takes the form :

$$\lambda_h v^h = -2\beta . \tag{3.6}$$

Therefore, in view of the above equation the formula (3.5) takes the form :

$$\lambda_s v^h{}_{(h)(j)} - \lambda_j v^h{}_{(h)(s)} - 2\beta H^h{}_{hsj} = 0 . \tag{3.7}$$

Contracting the latter part of (2.3) with respect to the indices i and j, we obtain

$$v^h{}_{(h)} = n\beta + \phi_h v^h = n\beta . \tag{3.8}$$

Differentiating the above equation covariantly with respect to x^m, we have

$$v^h{}_{(h)(m)} = n\beta_m . \tag{3.9}$$

Substituting the above equation into the left hand side of (3.7), we get

$$2\beta H^h{}_{hsj} = n(\lambda_s \beta_j - \lambda_j \beta_s) . \tag{3.10}$$

Equating the two equations, (2.22) and (3.10), we obtain

$$\begin{aligned} n(\lambda_s \beta_j - \lambda_j \beta_s) &= (\lambda_s + \phi_s) H^h{}_{hjm} v^m - (\lambda_j + \phi_j) H^h{}_{hsm} v^m \\ &= (\lambda_s + \phi_s) v^h{}_{(h)(j)} - (\lambda_j + \phi_j) v^h{}_{(h)(s)} \\ &= n(\lambda_s + \phi_s)\beta_j - n(\lambda_j + \phi_j)\beta_s \end{aligned} \tag{3.11}$$

or

$$\beta_j \phi_s - \beta_s \phi_j = 0 , \tag{3.12}$$

where we have used the equations (2.1) and (3.9) in the process of calculation. Consequently, there exists a function $\mu = \mu(x)$ such that

$$\phi_h = \mu\beta_h . \tag{3.13}$$

Making a cyclic interchange in the (3.4) with respect to the indices h, s and j, we get two more similar results. Adding the two equations thus obtained with (3.4), we have

$$\lambda_s (v^i{}_{(h)(j)} - v^i{}_{(j)(h)}) + \lambda_h (v^i{}_{(j)(s)} - v^i{}_{(s)(j)}) + \lambda_j (v^i{}_{(s)(h)} - v^i{}_{(h)(s)}) = 0 , \tag{3.14}$$

where we have used the Bianchi identity (1.5) during the process of calculation.

In view of the equation (2.4), the above relation reduces to

$$\begin{aligned} &\lambda_s (\beta\phi_h \delta^i_j + \beta_j \delta^i_h - \beta\phi_j \delta^i_h - \beta_h \delta^i_j) + \lambda_h (\beta\phi_j \delta^i_s + \beta_s \delta^i_j - \beta\phi_s \delta^i_j - \beta_j \delta^i_s) \\ &\quad + \lambda_j (\beta\phi_s \delta^i_h + \beta_h \delta^i_s - \beta\phi_h \delta^i_s - \beta_s \delta^i_h) = 0 , \end{aligned} \tag{3.15}$$

where we have used $\phi_{h(j)} = \phi_{j(h)}$.

Contracting the above equation with respect to the indices i and h, we get

$$(n-2)\{\lambda_j \beta_s - \lambda_s \beta_j + \beta(\lambda_j \phi_s - \lambda_s \phi_j)\} = 0 . \tag{3.16}$$

Thus, for $n \geqq 3$, the above formula reduces to

$$\beta(\lambda_j \phi_s - \lambda_s \phi_j) + \lambda_j \beta_s - \lambda_s \beta_j = 0 . \tag{3.17}$$

Introducing (3.13) into the left hand side of the above equation, we have

$$(\beta\mu + 1)(\lambda_j \beta_s - \lambda_s \beta_j) = 0 . \tag{3.18}$$

Therefore, we get here the following two cases :

$$\text{a)} \quad \beta\mu = -1 \qquad \text{and} \qquad \text{b)} \quad \lambda_j \beta_s - \lambda_s \beta_j = 0 . \tag{3.19}$$

On substituting (3.19b) into the right hand side of the formula (3.10), we obtain

$$\beta H^h{}_{hsj} = 0 . \tag{3.20}$$

So we have here $\beta H^h{}_{hsj} v^j = 0$ for a non-zero vector v^j. But we have assumed $H^h{}_{hsj} v^j \neq 0$, therefore from this conclusion, we get $\beta = 0$, that is, $\phi_h v^h = 0$ yields $\beta = 0$.

4. Main Discourse. For the case of $\beta = 0$, the equation of motion (2.3) takes the form :

$$\bar{x}^i = x^i + v^i(x)\, dt , \quad v^i{}_{(j)} = \phi_j(x)\, v^i , \tag{4.1}$$

but such a case was investigated by the author in [6].

Hence, we have to discuss, from now, remained case where (3.19a) holds good. In such a case, the motion to be studied becomes

$$\bar{x}^i = x^i + v^i(x)\, dt , \quad v^i{}_{(j)} = -\frac{1}{\beta} \beta_j v^i + \beta \delta^i{}_j . \tag{4.2}$$

Here, now let us introduce a contravariant vector η^i such that

$$\eta^i = \beta v^i . \tag{4.3}$$

Thus, we can now consider another infinitesimal point transformation

$$\bar{x}^i = x^i + \eta^i(x)\, dt . \tag{4.4}$$

Then, η^i becomes to define a special concircular field, for, we have

$$\eta^i{}_{(j)} = \beta_j v^i + \beta v^i{}_{(j)} . \tag{4.5}$$

Introducing the latter part of (4.2) into the above equation, we obtain

$$\eta^i_{(j)} = \beta_j v^i + \beta\left(-\frac{1}{\beta}\beta_j v^i + \beta\delta^i_j\right) = \beta^2 \delta^i_j . \tag{4.6}$$

Thus, the system (4.2) may be reduced to a special concircular form :

$$\bar{x}^i = x^i + \eta^2_i(x)\, dt , \quad \eta^i_{(j)} = \beta^2 \delta^i_j , \quad \beta \neq 0 . \tag{4.7}$$

This case has been deeply studied by the author in [5].

In concluding this paper, we shall speak of an exceptional case, where

$$H^h_{hsm} v^m = 0 . \tag{4.8}$$

If it will be the case, we can get from the equation (3.8), the following relation :

$$v^h_{(h)(s)} = (\phi_h v^h)_{(s)} + n\beta_s = 0 . \tag{4.9}$$

In view of the lemma (2.2), the above equation reduces to

$$\beta_s = 0 \qquad \text{or} \qquad \beta = \text{Const.} \tag{4.10}$$

By virtue of the above formula, the equation (2.27) takes the form :

$$v^h H^i_{hji} = (n-1)\beta\phi_j . \tag{4.11}$$

Differentiating (4.11) covariantly with respect to x^m and noting the equations (1.11) and (2.3), we have

$$v^h H^i_{hji}(\lambda_m + \phi_n) + \beta H^h_{mjh} = (n-1)\beta\phi_{j(m)} . \tag{4.12}$$

Introducing the left hand side of (4.11) into (4.12), we get

$$(n-1)\beta\phi_j(\lambda_m + \phi_m) + \beta H^h_{mjh} = (n-1)\beta\phi_{j(m)} . \tag{4.13}$$

In view of the equation (1.7) transvecting the above relation by v^j and summing over the index j, we obtain

$$(n-1)\beta\phi_j v^j(\lambda_m + \phi_m) - \beta v^h_{(m)(h)} = (n-1)\beta\phi_{j(m)} v^j . \tag{4.14}$$

For a non-vanishing β, the above equation yields

$$(n-1)\phi_j v^j(\lambda_m + \phi_m) - v^h_{(m)(h)} = (n-1)\phi_{j(m)} v^j . \tag{4.15}$$

By virtue of the equations (2.13), (2.16) and (4.10), the above equation reduces to

$$\lambda_m \phi_j v^j = \phi_{j(m)} v^j - \beta\phi_m - \frac{1}{(n-1)}\beta_m - \beta_m \beta_j v^j = -2(\phi_j v^j + \beta)\phi_m . \tag{4.16}$$

According to the lemma (2.2) and the equation (4.10), $\gamma = \phi_j v^j + \beta$ and $\mu = \phi_j v^j$ means a constant respectively. And the equation (4.16) can be written like

$$\mu\lambda_m = -2\gamma\phi_m . \tag{4.17}$$

The case of $\mu = 0$ gives $\beta = 0$, so the point transformation becomes one of the special recurrent form:

$$\bar{x}^i = x^i + v^i(x)\, dt , \quad v^i_{(j)} = \phi_j v^i \ (\phi_j : \text{ gradient vector}). \tag{4.18}$$

Next, we consider that $\mu \neq 0$. For this case, λ_s becomes a gradient vector and £v $H^i{}_{hjk} = 0$ gives

$$\varepsilon^2 (\lambda_s v^s + 2\beta) H_{hj} - 2\beta (n-1) \lambda_h \lambda_j = 0 , \tag{4.19}$$

where

$$\varepsilon \overset{\text{def.}}{=\!=} -2\lambda/\mu . \tag{4.20}$$

But $\lambda_s v^s + 2\beta = 0$ or $\varepsilon = 0$ implies $\lambda_h = 0$. In non-zero case, we can write

$$H_{hj} = d\lambda_h \lambda_j \ \ (\text{d} = \text{non-zero const.}). \tag{4.21}$$

By virtue of the definition (1.11) differentiating covariantly with respect to x^m, we obtain

$$\lambda_m H_{hj} = d(\lambda_{h(m)} \lambda_j + \lambda_h \lambda_{j(m)}) . \tag{4.22}$$

Again differentiating the above equation covariantly with respect to x^s and using the relation (1.11), we get

$$\lambda_m \lambda_s H_{hj} + \lambda_{m(s)} H_{hj} = d(\lambda_{h(m)(s)} \lambda_j + \lambda_{h(m)} \lambda_{j(s)} + \\ + \lambda_{h(s)} \lambda_{j(m)} + \lambda_h \lambda_{j(m)(s)}) . \tag{4.23}$$

By virtue of the commutation formula (1.3) commutating the above equation with respect to the indices m and s, we have

$$\lambda_j \lambda_\gamma H^\gamma{}_{hms} + \lambda_h \lambda_\gamma H^\gamma{}_{jms} = 0 . \tag{4.24}$$

By virtue of the equations (2.1), (4.24) and the fact that $\varepsilon\, \phi_j = \lambda_j$, we can conclude

$$\lambda_h v^h \lambda_\gamma H^\gamma{}_{jms} = 0 . \tag{4.25}$$

Hence, we get $\lambda_h v^h = 0$ or $\lambda_\gamma H^\gamma{}_{jms} = 0$. If $\lambda_h v^h = 0$, holds, by virtue of (4.17), $\lambda_h = 0$ and if $\lambda_\gamma H^\gamma{}_{jms} = 0$ holds, λ_s means a parallel vector field. Therefore, in view of the equations (4.21) and (4.22), we can see

$$\lambda_s = 0 . \tag{4.26}$$

Thus, we have:

Theorem (4.1). If a non-symmetric PRF_n-space (that is, $\lambda_j \neq 0$) admits a projective affine motion of concircular form

$$\bar{x}^i = x^i + v^i(x)\, dt\,, \quad v^i_{(j)} = \phi_j(x)\, v^i + \beta\delta^i_j$$

$$\phi = \text{gradient vector}, \quad \beta = \text{scalar function}$$

then $\beta(x)$ must be zero function. Consequently the motion should be a recurrent form.

Theorem (4.2). If an n-dimensional PRF_n-space admits a non-recurrent concircular projective affine motion (2.3) with $\phi_j v^j$ non-zero constant, the space must be a projective symmetric space.

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ÖZET

Bu çalışmada

$$\bar{x}^i = x^i + v^i(x)\, dt,\; v^i_{(j)} = \beta\delta^i_j + \phi_j(x)\, v^i$$

formunda bir infinitezimal afin hareketin varlığı incelenmektedir. Burada $\beta(x)$ herhangi bir fonksiyonu, $\phi_j(x)$ te

$$\phi_j(x) = \phi_{(j)} = \partial_j \phi$$

ile tanımlanan bir gradiyent vektörü göstermektedir.