PROJECTIVE AFFINE MOTION IN A PRF$_n$-SPACE, IV

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In this paper an infinitesimal special projective affine motion in a PRF$_n$-space is studied.

1. Introduction. Let us consider an n-dimensional affinely connected and non flat Finsler space F_n [1][1]) having symmetric Berwald's connection coefficient $G^i_{jk}(x, \dot{x})$. The covariant derivative of any tensor field $T^i_j(x, \dot{x})$ in the sense of Berwald with respect to x^k is given by

$$T^i_{j(k)} = \partial_k T^i_j - \dot{\partial}_m T^i_j G^m_{\gamma k} \dot{x}^\gamma + T^s_j G^i_{sk} - T^i_s G^s_{jk} . \tag{1.1}$$

The commutation formula involving the above covariant derivative is given by

$$2 T^i_{j[(h)(k)]} = - \dot{\partial}_\gamma T^i_j H^\gamma_{hk} + T^s_j H^i_{shk} - T^i_s H^s_{jhk}\,^{2), 3)}, \tag{1.2}$$

where

$$H^i_{hjk}(x, \dot{x}) \overset{\text{def.}}{=} 2 \{\partial_{[k} G^i_{j]h} - G^i_{\gamma h[j} G^\gamma_{k]} + G^\gamma_{h[j} G^i_{k]\gamma}\} \tag{1.3}$$

is called Berwald's curvature tensor field and satisfies the following relation [1]:

$$H^i_{hji} = H_{hj}, \tag{1.4}$$

$$H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0 \tag{1.5}$$

and

$$H^i_{hjk} = - H^i_{hkj} . \tag{1.6}$$

Let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\, dt , \tag{1.7}$$

1) Numbers in square brackets refer to the references at the end of the paper.

2) $2 A_{[hk]} = A_{hk} - A_{kh}$

3) $\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$ and $\partial_i \equiv \partial/\partial x^i$.

where $v^i(x)$ is any vector field and dt is an infinitesimal point constant. In view of the above point transformation and Berwald's covariant derivative, we have the following well known results [2] :

$$£v\, T^i_j = T^i_{j(h)}\, v^h - T^h_j\, v^i_{(h)} + T^i_h\, v^h_{(j)} + \dot{\partial}_h\, T^i_j\, v^h_{(\gamma)}\, \dot{x}^\gamma \tag{1.8}$$

and

$$£v\, G^i_{jk} = v^i_{(j)(k)} - H^i_{jkh}\, v^h + G^i_{sjk}\, v^s_{(\gamma)}\, \dot{x}^\gamma , \tag{1.9}$$

where $G^i_{sjk} \equiv \dot{\partial}_s\, G^i_{jk}$ and £v denotes Lie-operator. We have also the following well known commutation formula :

$$£v\, (T^i_{j(h)}) - (£v\, T^i_j)_{(h)} = 0 . \tag{1.10}$$

In an F_n-space, if the Berwald's curvature tensor satisfies the relation

$$H^i_{hjk(s)} = \lambda_s\, H^i_{hjk} , \tag{1.11}$$

where λ_s is any covariant vector field, then the space is called projective recurrent Finsler space of first order or PRF_n-space.

The gradient vector of PRF_n-space is given by

$$\lambda_s = \frac{1}{\lambda}\, \partial_s\, \lambda . \tag{1.12}$$

In a previous paper we have concluded as follows :

If a PRF_n-space admits an infinitesimal affine motion $\bar{x}^i = x^i + v^i(x)\, dt$, we have $\lambda_s\, v^s = 0$, say the function λ being a Lie invariant one.

This fact will be revised concretely and eloquently in the following manner: It is well known that when F_n admits a projective affine motion characterized by

$$£v\, G^i_{jk} = 0 \tag{1.13}$$

then the two operations, that is, Lie derivation £v and covariant differentiation (j) are commutative with each other. Then let us operate £v to the both sides of the fundamental and starting condition (1.11) of PRF_n-space, we get

$$(£v\, H^i_{hjk})_{(s)} = (£v\, \lambda_s)\, H^i_{hjk} + \lambda_s\, £v\, H^i_{hjk} , \tag{1.14}$$

where we have used the equation (1.10).

The integrability condition of the projective affine motion (1.13) is given by

$$£v\, H^i_{hjk} = 0 . \tag{1.15}$$

In this way, with the help of the above formula, the equation (1.14) takes the form

$$(\pounds v\, \lambda_s)\, H^i{}_{hjk} = 0\,, \tag{1.16}$$

which for the non-flatness property of the space reduces to

$$\pounds v\, \lambda_s = 0\,. \tag{1.17}$$

However, from the above equation in the present gradient case, we have

$$\lambda_{s(m)} = \lambda_{m(s)}\,. \tag{1.18}$$

The above formula can be also written as

$$(\lambda_s\, v^s)_{(m)} = 0\,, \tag{1.19}$$

i.e.

$$\lambda_s\, v^s = c \quad \text{or} \quad \pounds v\, \lambda = c, \quad c = \text{arbitrary const.} \tag{1.20}$$

2. Some **Appendices** to the Recurrent **Motion.** The present author has already studies on the projective affine motion in PRF_n-space [6]. In the same paper, I have pursued the concrete form of such a motion and obtained the following two cases :

$$\text{a)}\quad \lambda_h + \psi_h = 0 \qquad \text{and} \qquad \text{b)}\quad H_{hs}\, v^s = 0\,. \tag{2.1}$$

But, I have, throughout the sections, only taken up the former case. And, in fact, differentiating (2.1b) covariantly with respect to x^m we can see

$$(\lambda_m + \psi_m)\, H_{hs}\, v^s = 0\,. \tag{2.2}$$

Thus, $(\lambda_m + \psi_m)$ may be taken arbitrarily. In this meaning, the former case means only a special case contained in (2.1b). Thus, we know that, in order to discuss generally the projective recurrent affine motion in a PRF_n-space, we must take up the case where (2.1b) holds good. We shall consider this general case in the following lines. In view of the equation (1.4), the relation (2.1b) can also be written as

$$H^i{}_{hsi}\, v^s = 0\,. \tag{2.3}$$

Contracting the Bianchi's first identity (1.5) for $H^i{}_{hjk}(x, \dot{x})$ with respect to the indices i and k, we obtain

$$H^i{}_{hji} = -\, H^i{}_{jih} - H^i{}_{ihj}\,. \tag{2.4}$$

Transvecting the above result by v^j and taking care of the equation (2.3) we find

$$H^i{}_{jih}\,v^j + H^i{}_{ihj}\,v^j = 0\,. \tag{2.5}$$

By virtue of the equations (1.6), (1.9) and (1.13), the last formula reduces to

$$-H^i{}_{jhi}\,v^j + v^i{}_{(i)\,(h)} = 0\,. \tag{2.6}$$

Being $v^i{}_{(h)} = \psi_h\,v^i$, we have

$$v^i{}_{(i)\,(j)} = (\psi_i\,v^s)_{(j)}\,. \tag{2.7}$$

In view of the above relation, the formula (2.6) can be written as

$$H_{jh}\,v^j = (\psi_s\,v^s)_{(h)}\,. \tag{2.8}$$

On the other hand we have the following relation :

$$v^i{}_{(h)} = \psi_h\,v^i\,. \tag{2.9}$$

In this way, taking the covariant derivative of (2.8) with respect to x^m and using the equations (1.11) and (2.9), we get

$$(\lambda_m + \psi_m)\,H_{jh}\,v^j = (\psi_s\,v^s)_{(h)\,(m)}\,. \tag{2.10}$$

Now, eliminating the term $H_{jh}\,v^j$ with the help of the equations (2.8) and (2.10), we obtain

$$(\lambda_m + \psi_m)\,(\psi_s\,v^s)_{(h)} = (\psi_s\,v^s)_{(h)\,(m)}\,. \tag{2.11}$$

By virtue of the symmetric properties of the connection coefficient $G^i{}_{jk}$ we can conclude

$$(\psi_s\,v^s)_{(h)\,(m)} = (\psi_s\,v^s)_{(m)\,(h)}\,. \tag{2.12}$$

Consequently with the help of the equations (2.11) and (2.12), we can have

$$(\lambda_h + \psi_h)\,(\psi_s\,v^s)_{(k)} = (\lambda_k + \psi_k)\,(\psi_s\,v^s)_{(h)}\,. \tag{2.13}$$

In view of the equations (2.1b) and (2.8), we can construct

$$(\psi_s\,v^s)_{(h)}\,v^h = (H_{mh}\,v^m)\,v^h = (H_{mh}\,v^h)\,v^m = 0\,. \tag{2.14}$$

Thus, transvecting the equation (2.13) by v^k and taking care of last formula, we have

$$(\lambda_k + \psi_k)\,v^k\,(\psi_s\,v^s)_{(h)} = 0\,. \tag{2.15}$$

In this way, we can obtain here two cases :

$$\text{a)}\quad (\lambda_k + \psi_k)\,v^k = 0 \quad \text{or} \quad \text{b)}\quad \psi_s\,v^s = \text{const.} \tag{2.16}$$

Thus, from the last equation, we can say :

If a PRF$_n$-space admits a projective affine motion of recurrent form then there exist the following two interesting fields :

$$\begin{aligned} &\text{(i)} \quad \text{A case of } (\lambda_h + \psi_h)\, v^h = 0 \\ &\text{(ii)} \quad \text{A case of } \psi_h\, v^h = \text{constant.} \end{aligned} \tag{2.17}$$

If we take especially $\psi_h = -\lambda_h$ so as to satisfy (2.16a), we can develop the existence theory written in [6]. Thus, we know that there are able to exist three categories of projective recurrent affine motions such as

(A) $\bar{x}^i = x^i + v^i(x)\, dt, \; v^i_{(j)} = \psi_j v^i, \; \psi_h v^h = \text{const.},$

(B) $\bar{x}^i = x^i + v^i(x)\, dt, \; v^i_{(j)} = \psi_j v^i, \; (\psi_h + \lambda_h)\, v^h = 0,$

(C) $\bar{x}^i = x^i + v^i(x)\, dt, \; v^i_{(j)} = \psi_j v^i, \; (\psi_h + \lambda_h) = 0$

being derived from (B) formally. In Chapter 5, we shall try to make the concrete relation existing between (B) and (C) clear.

3. Study on the Case (A).

§1. **Necessary conditions.** In this case we have

$$\psi_h v^h = \text{const.} \tag{3.1}$$

Differentiating this condition covariantly with respect to x^s and using the condition (A), we find

$$\psi_{h(s)} v^h + \psi_h \psi_s v^h = 0 . \tag{3.2}$$

In view of the equations (1.9), (1.13) and the condition (A), we can get

$$H^i{}_{jkh} v^h = (\psi_j v^i)_{(k)} = \psi_{j(k)} v^i + \psi_j \psi_k v^i . \tag{3.3}$$

By virtue of the fact $H^i{}_{hjk} v^j v^k = 0$, transvecting the last formula by v^k, we obtain

$$\psi_{j(k)} v^k + \psi_j \psi_k v^k = 0 . \tag{3.4}$$

Thus comparing the last result with (3.2), we find

$$(\psi_{j(k)} - \psi_{k(j)})\, v^k = 0 . \tag{3.5}$$

In an affinely connected F_n the second Bianchi's identity for Berwald's curvature tensor field $H^i{}_{hjk}(x, \dot{x})$ takes the form

$$H^i{}_{hjk(s)} + H^i{}_{hks(j)} + H^i{}_{hsj(k)} = 0 . \tag{3.6}$$

By virtue of the definition (1.11) the last equation can be written as

$$\lambda_s H^i_{hjk} v^k + H^i_{hks} v^k \lambda_j + H^i_{hsj} \lambda_k v^k = 0 . \qquad (3.7)$$

Interchanging the indices h, s and j cyclically of the above equation, we get two more similar relations. Adding all these equations with (3.7) and taking care of the formulae (1.5) and (1.6), we have

$$(H^i_{hjk} v^k - H^i_{jhk} v^k) \lambda_s + (H^i_{shk} v^k - H^i_{hsk} v^k) \lambda_j + \\ + (H^i_{jsk} v^k - H^i_{sjk} v^k) \lambda_h = 0 . \qquad (3.8)$$

In view of the equations (1.9) and (1.13), the above formula reduces to

$$\lambda_s (v^i_{(h)(j)} - v^i_{(j)(h)}) + \lambda_j (v^i_{(s)(h)} - v^i_{(h)(s)}) + \\ + \lambda_h (v^i_{(j)(s)} - v^i_{(s)(j)}) = 0 . \qquad (3.9)$$

By virtue of the conditions (a) the last equation can be written as

$$\lambda_s (\psi_{h(j)} - \psi_{j(h)}) v^i + \lambda_j (\psi_{s(h)} - \psi_{h(s)}) v^i + \lambda_h (\psi_{j(s)} - \psi_{s(j)}) v^i = 0 . \quad (3.10)$$

Now contracting the above relation with respect to the indices i and h, we have

$$v^h \lambda_h (\psi_{j(s)} - \psi_{s(j)}) = 0, \qquad (3.11)$$

where we have used (3.5).

With the help of the last equations, we find here the following conditions :

$$\text{(i)} \quad \lambda_h v^h = 0 \quad \text{or} \quad \text{(ii)} \quad \psi_{j(s)} = \psi_{s(j)} . \qquad (3.12)$$

Thus we can state :

When a PRF_n-space admits a projective affine motion of recurrent form the condition (3.1) is necessitated. But, in such a case, we have the following :

(A)′ $\lambda_h v^h = 0$

(B)′ Defining vector $\psi_j(x)$ of the motion should be gradient vector.

§2. **The case of ψ_j = gradient vector.**

In such a case, the form of the motion is given by

$$\bar{x}^i = x^i + v^i(x)\, dt, \; v^i_{(j)} = \psi_j v^i , \; \psi_j(x) = \text{gradient vector}. \qquad (3.13)$$

On the other hand, let us introduce the following commutator :

$$M^i_{jk} \overset{\text{def.}}{=} (v^i_{(j)(k)} - v^i_{(k)(j)}) . \qquad (3.14)$$

By virtue of the equation (3.13), the above formula takes the form

$$M^i_{jk} = \psi_{j(k)} v^i + \psi_j v^i_{(k)} - \psi_{k(j)} v^i + \psi_k v^i_{(j)} = 0 , \qquad (3.15)$$

where we have used the gradient property of ψ_j, i.e.

$$\psi_{j(k)} = \psi_{k(j)} \,. \tag{3.16}$$

Furthermore, in view of the commutation formula (1.2) and the relation (3.15), the equation (3.14) may be replaced by

$$H^i{}_{hjk}\, v^h = 0 \,. \tag{3.17}$$

By the general theory of fields of parallel vectors [4], the last equation shows that $v^i(x)$ determines a field of parallel vector. Consequently the motion characterized by (3.13) is not a pure recurrent motion, but a contra-motion in the general sense.

Now if (3.13) denotes exactly a projective affine motion, it has to satisfy the integrability condition

$$\begin{aligned} £v\, H^i{}_{hjk} = {} & H^i{}_{hjk}\, v^s \lambda_s - H^s{}_{hjk}\, v^i{}_{(s)} + H^i{}_{sjk}\, v^s{}_{(h)} + H^i{}_{hsk}\, v^s{}_{(j)} + \\ & + H^i{}_{hjs}\, v^s{}_{(k)} + \dot{\partial}_s H^i{}_{hjk}\, v^s{}_{(\gamma)}\, \dot{x}^\gamma = 0 \end{aligned} \tag{3.18}$$

of the equation (3.13).

By virtue of the equation (3.13) the last formula takes the form

$$\lambda_s v^s H^i{}_{hjk} - H^s{}_{hjk}\, \psi_s v^i + H^i{}_{sjk}\, \psi_h v^s + H^i{}_{hsk}\, \psi_j v^s + H^i{}_{hjs}\, \psi_k v^s = 0 \,. \tag{3.19}$$

But, in view of the equation (3.17), the above relation can be written like

$$\lambda_s v^s H^i{}_{hjk} - H^s{}_{hjk}\, \psi_s v^i + H^i{}_{hsk}\, \psi_j v^s + H^i{}_{hjs}\, v^s \psi_k = 0 \,. \tag{3.20}$$

Contracting the above formula with respect to the indices i and k, we find

$$\lambda_s v^s H_{hj} + \psi_j H_{hs} v^s = 0 \,, \tag{3.21}$$

where we have used (1.4).

Now, our present theory is based on the condition (2.1b). Therefore, the last relation yields

$$\lambda_s v^s H_{hj} = 0 \,. \tag{3.22}$$

Being $\lambda_s v^s \neq 0$, the last formula can also be written as

$$H_{hj} = 0 \,. \tag{3.23}$$

From the above discussions, first of all, we have :

Theorem 3.1. If a PRF_n-space admits a special recurrent motion or contra-motion in the general sense of the form derived from (A) :

$$\bar{x}^i = x^i + v^i(x)\, dt, \;\; v^i{}_{(j)} = \psi_j v^i \,, \;\; \psi_{j(k)} = \psi_{k(j)}$$

with $\lambda_s v^s \neq 0$, the space has a property given by the equation (3.23).

4. Study on the case (B). For this case the starting condition is given by

$$(\psi_h + \lambda_h)\, v^h = 0\,. \tag{4.1}$$

Transvecting the integrability condition (3.19) of projective affine motion by v^k and taking care of the condition (B) and (4.1), we get

$$\begin{aligned} &-\psi_s v^s (\psi_{h(j)} v^i + \psi_h \psi_j v^i) - \psi_s v^i (\psi_{h(j)} v^s + \psi_h \psi_j v^s) + \\ &+ \psi_h v^s (\psi_{s(j)} v^i + \psi_s \psi_j v^i) + \psi_j v^s (\psi_{h(s)} v^i + \psi_h \psi_s v^i) + \\ &+ \psi_k v^k (\psi_{h(j)} v^i + \psi_h \psi_j v^i) = 0\,, \end{aligned} \tag{4.2}$$

where we have also used

$$H^i{}_{hjk} v^k = \psi_{(h)(j)} - G^i{}_{shj} v^s{}_{(\gamma)} \dot{x}^\gamma = (\psi_h v^i)_{(j)} = \psi_{h(j)} v^i + \psi_h \psi_j v^i\,. \tag{4.3}$$

After little simplification the formula (4.2) may be replaced by

$$-\psi_s v^s \psi_{h(j)} v^i + \psi_h v^s \psi_{s(j)} v^i + \psi_j v^s \psi_{h(s)} v^i + \psi_k \psi_h \psi_j v^k v^i = 0\,. \tag{4.4}$$

Commutating the last formula with respect to the indices h and j, we find

$$-\psi_s v^s (\psi_{h(j)} - \psi_{j(h)}) + \psi_h (\psi_{s(j)} - \psi_{j(s)}) v^s + \psi_j (\psi_{h(s)} - \psi_{s(h)}) v^s = 0\,, \tag{4.5}$$

where we have neglected the non-vanishing $v^i(x)$.

Contracting the relation (3.10) with respect to the indices i and s and noting the condition (4.1), we obtain

$$-\psi_s v^s (\psi_{h(j)} - \psi_{j(h)}) - \lambda_j (\psi_{h(s)} - \psi_{s(h)}) v^s - \lambda_h (\psi_{s(j)} - \psi_{j(s)}) v^s = 0\,. \tag{4.6}$$

On the other hand subtracting the above formula from (4.5), we find

$$(\psi_j + \lambda_j)(\psi_{h(s)} - \psi_{s(h)}) v^s + (\psi_h + \lambda_h)(\psi_{s(j)} - \psi_{j(s)}) v^s = 0\,. \tag{4.7}$$

Let us put

$$E_s = (\psi_{h(s)} - \psi_{s(h)})\, v^h\,. \tag{4.8}$$

Now when and only when $E_s \neq 0$, there exists a suitable proportional scalar function $\alpha(x)$ such that

$$(\psi_s + \lambda_s) = \alpha(x)\, E_s\,, \tag{4.9}$$

where E_s satisfies the relation

$$E_s v^s = 0\,. \tag{4.10}$$

By virtue of the equations (2.1b) and (4.8), we can have

$$\begin{aligned} E_s v^s &= \{(\psi_h v^h)_{(s)} - \psi_h v^h{}_{(s)} - (\psi_s v^h)_{(h)} + \psi_s v^h{}_{(h)}\} v^s \\ &= \{v^h{}_{(h)(s)} - \psi_h \psi_s v^h - v^h{}_{(s)(h)} + \psi_s \psi_h v^h\} v^s \\ &= (v^h{}_{(h)(s)} - v^h{}_{(s)(h)}) v^s = (v^m H^h{}_{msl}) v^h \\ &= (H_{ms} v^m) v^s = (H_{ms} v^s) v^m = 0 , \end{aligned} \tag{4.11}$$

where we have used (1.4) and the commutation formula (1.2).

This completes the proof of the condition (4.10). Being E_s given, as above, by

$$E_s = H^h{}_{msh} v^m = H_{ms} v^m , \tag{4.12}$$

we can derive

$$E_{s(k)} = (\lambda_k + \psi_k) E_s . \tag{4.13}$$

In view of the definition (4.9), the last result can be written as

$$E_{s(k)} = \alpha E_k E_s . \tag{4.14}$$

Thus, by virtue of the equations (4.1) and (4.10), the last formula yields

$$E_{s(k)} v^k = 0 . \tag{4.15}$$

On the other hand, with the help of the equations (2.8) and (4.12), we can find

$$E_s v^s = (\psi_m v^m)_{(s)} v^s . \tag{4.16}$$

By virtue of the identity $(\psi_m v^m)_{(s)} v^s = 0$, the above formula reduces to (4.10). Furthermore, for a projective affine motion, we have always

$$£v \lambda_s = \lambda_{s(m)} v^m + \lambda_m v^m{}_{(s)} = 0 \tag{4.17}$$

or

$$£v \lambda_s = (\lambda_{s(m)} + \lambda_m \psi_s) v^m = 0 . \tag{4.18}$$

Introducing the value of λ_s from the formula (4.9) into the above equation, we get

$$\begin{aligned} (\alpha E_s - \psi_s)_{(m)} v^m + (\alpha E_m - \psi_m) \psi_s v^m &= \alpha_{(m)} E_s v^m + \alpha E_{s(m)} v^m - \\ &\quad - \psi_{s(m)} v^m + \alpha E_m \psi_s v^m - \psi_m \psi_s v^m = 0 \end{aligned}$$

$$E_s £v \alpha(x) + \alpha E_{s(m)} v^m - (\psi_{s(m)} v^m + \psi_m \psi_s v^m) + \alpha E_m \psi_s v^m = 0 . \tag{4.19}$$

Hereupon, if we take care of the equations (3.3), (4.10) and (4.15), we can obtain a remarkable property :

$$E_s £v \alpha(x) = 0 . \tag{4.20}$$

From the above equation, in case of (4.1), we find

$$\pounds v \, \alpha(x) = 0 . \tag{4.21}$$

5. Stand Point of the Paper II (Recurrent Case). The author has already studied the existence of projective recurrent affine motion in a **PRF**$_n$-space in [6]. In that paper the basic property for this condition was

$$(\psi_h + \lambda_h) = 0 . \tag{5.1}$$

Here, we shall write down a diagram of the main course discussed in that paper [6]:

$$(\psi_h + \lambda_h) = 0 \rightarrow H_{hs} v^s = 0 \rightarrow H_{hs} = \lambda_h p_s \quad \text{or} \quad \lambda_h v^h = 0 , \tag{5.2}$$

where p_s means a suitable gradient vector defined by

$$\rho_s \equiv \frac{1}{\rho} \rho_{(s)} = \frac{1}{\rho} \partial_s \rho , \quad \rho = p(x) \tag{5.3}$$

and also satisfies the following relations:

$$\text{a)} \quad p_s v^s = 0 \qquad \text{and} \qquad \text{b)} \quad \rho_{h(s)} = p_h p_s . \tag{5.4}$$

Now, we have introduced a vector E_s and this has taken the form

$$E_s = (\psi_h v^h)_{(s)} = \epsilon_s , \tag{5.5}$$

where

$$\epsilon \overset{\text{def.}}{=} \psi_h v^h . \tag{5.6}$$

Hence E_s denotes a gradient vector.

In view of the equations (4.10), (4.14), (5.5) and the gradience property of $p_h(x)$ and E_h we can conclude that E_h is a vector very similar to ρ_h.

Now let us remember the assumption of the integrability conditions of projective affine motion:

$$\begin{aligned} \pounds v \, H^i{}_{hjk} = & - \psi_s v^s H^i{}_{hjk} - \psi_s v^i H^i{}_{hjk} + \psi_h v^s H^i{}_{sjk} + \\ & + \psi_j v^s H^i{}_{hsk} + \psi_k v^s H^i{}_{hjs} = 0 . \end{aligned} \tag{5.7}$$

Contracting the above equation with respect to the indices i and k we find

$$- \psi_s v^s H_{hj} + \psi_h v^s H_{sj} = 0 , \tag{5.8}$$

where we have used the equations (1.4), (2.1b) and (4.1). By virtue of the equation (4.12) the last formula reduces to

$$\psi_s v^s H_{hj} = \psi_h E_j . \tag{5.9}$$

Now in the present case, $\psi_s v^s$ denotes a non-constant scalar function $\in (x)$. Then we shall assume the resolvability of H_{hj} of the form

$$H_{hj} = \lambda_h E_j , \quad \lambda_h \equiv \frac{1}{\in} \psi_h . \tag{5.10}$$

Furthermore substituting the value of λ_h from the above equation into the formula (4.1), we get

$$\left(\frac{1}{\in} \psi_h + \psi_h \right) v^h = 0 \tag{5.11}$$

or

$$(1 + \in) \psi_h v^h = 0 . \tag{5.12}$$

Being $(1 + \in) \neq 0$, so we have $\psi_h v^h = 0$, this contradicts our assumption. In the second step, we try to resolve H_{hj} in the form

$$H_{hj} = \lambda_h \eta_j , \tag{5.13}$$

where

$$\text{a)} \quad \eta_j \stackrel{\text{def.}}{=\!=} - \frac{1}{\in} E_j \quad \text{and} \quad \text{b)} \quad \eta_{j(s)} = \eta_j \eta_s . \tag{5.14}$$

With the help of the equations (5.10) and (5.13), comparing the forms of H_{hj}, we have

$$H_{hj} = \frac{1}{\in} \psi_h E_j = \lambda_h \left(- \frac{1}{\in} E_j \right) \tag{5.15}$$

or

$$\frac{1}{\in} (\psi_h + \lambda_h) E_j \tag{5.16}$$

say, we are trying to have $(\psi_h + \lambda_h) = 0$ for a non-zero E_j. That is, we want to get a set which we have showed in the diagram :

$$v^i{}_{(j)} = - \lambda_j v^i , \quad H_{hj} = \lambda_h \eta_j , \quad \eta_{j(s)} = \eta_j \eta_s . \tag{5.17}$$

If this is possible because of (4.10), we obtain

$$\eta_h v^h = 0 . \tag{5.18}$$

Such a case has been really the main part in the paper [6] in question. We shall assume this fact and seek for a necessary condition for this case hidden in our theory.

The formula (5.14a) can also be rewritten as

$$E_j = -\epsilon\,\eta_j\,. \tag{5.19}$$

Differentiating the above formula covariantly with respect to x^s and taking care of the equations (5.4), (5.14b) and (5.19) itself, we find

$$E_{j(s)} = -E_s\,\eta_j - \epsilon\,\eta_j\,\eta_s = -E_s\,\eta_j - (\epsilon\,\eta_s)\,\eta_j = -E_s\,\eta_j + E_s\,\eta_j = 0\,. \tag{5.20}$$

On the other hand, from (4.14), we can say that in order that the present theory contains the main theory of the paper [6] it is necessary that we have $\alpha = 0$, furthermore, $\alpha = 0$ satisfies (4.21) certainly, hence it is reasonable to consider such a case.

Conversely, if $\alpha = 0$, we get $(\psi_h + \lambda_j) = 0$ and the paper [6] comes in front of us.

Thus we can state here :

Theorem 5.1. In order that a projective recurrent affine motion, in a PRF_n-space

$$\bar{x}^i = x^i + v^i(x)\,dt\,,\quad v^i_{(h)} = \psi_h\,v^i\,,$$

admitted under assumptions (2.1b) and (4.1) becomes a main motion of the same kind :

$$\bar{x}^i = x^i + v^i(x)\,dt\,,\quad v^i_{(h)} = -\lambda_h\,v^i\,,\quad H_{hj} = \lambda_h\,\eta_j\,,\quad \eta_{j(s)} = \eta_j\,\eta_s$$

it is necessary and sufficient that we have $\alpha(x) = 0$, where $\alpha(x)$ denotes a proportional factor such that

$$\lambda_h + \psi_h = \alpha(x)\,H_{sh}\,v^h \quad \text{and} \quad \pounds v\,\alpha(x) = 0\,.$$

6. Appendices to Chapter 3. At the end of Chapter 3, the author has omitted to say some conclusions. In case of $\psi_j =$ non-gradient vector, we can state Theorem 6.1. When $\psi_h\,v^h =$ constant and ψ_j is not a gradient vector, the PRF_n-space is able to admit a projective recurrent motion of the form

$$\bar{x}^i = x^i + v^i(x)\,dt\,,\; v^i_{(h)} = \psi_h\,v^i \quad \text{with} \quad \lambda_h\,v^h = 0\,.$$

All process of calculation developed in §1 and §2 of Chapter 3 holds similarly when ψ_j is replaced by $-\lambda_j$ formally. Hence we have :

Theorem 6.1. In the existence theory of special recurrent affine motion of the form

$$\bar{x}^i = x^i + v^i(x)\,dt\,,\quad v^i_{(h)} + \lambda_h\,v^i = 0\,, \tag{6.1}$$

if we put an additional condition

$$\lambda_h v^h = \text{const.} \tag{6.2}$$

we can obtain naturally the following two independent categories of the motion

$$\bar{x}^i = x^i + v^i(x)\,dt\,, \quad v^i_{(h)} + \lambda_h v^i = 0 \tag{6.3}$$

satisfying

$$\lambda_h v^h \neq 0\,,\ \lambda_{h(s)} = \lambda_{s(h)} \quad \text{and} \quad H_{hj} = 0\,, \tag{6.4}$$

$$\bar{x}^i = x^i + v^i(x)\,dt\,, \quad v^i_{(h)} + \lambda_h v^i = 0 \tag{6.5}$$

satisfying

$$\lambda_h v^h = 0\,. \tag{6.6}$$

These two results appeared actually in the paper [6]. The former occurs when $p_s = 0$. In the forthcoming paper we shall go into details about these two motions and revise some points in the paper [6].

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ÖZET

Bu çalışmada bir PRF_n - uzayındaki özel bir infinitezimal hareket incelenmektedir.