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A NOTE ON COMPACT CONVEX SETS WITH EQUAL SUPPORT PROPERTY

Ş. ALPAY[1]

The background material for the following can be found in [1]. It is well known that a compact convex set X is a Choquet simplex if and only if each x in X has a unique α-maximal representing measure where α denotes the Choquet's ordering on the set $M^+(X)$. In this paper we study compact convex sets with a certain property, called the equal support property [2].

1. Preliminaries and notation. The expression "compact convex set" will always refer to a non-empty compact convex subset of a locally convex Hausdorff linear space. We shall use the following symbols :

X_e : the set of extreme points of X

$C(X)$: the Banach space of continuous real-valued functions on X

$A(X)$: the Banach space of continuous affine real-valued functions on X.

We shall denote by $M(X)$ the Banach space of all signed Radon measures on X, and by $M_1^+(X)$ the w^*-compact convex set of normalized positive Radon measures on X.

A signed measure μ on X is said to be a boundary measure if $|\mu|$ is maximal in Choquet's ordering of positive measures [1]. We say that a measure μ on X represents the point x of X (or x is the barycentre of (μ)) if $f(x) = \int_X f(y)\, d\mu(y)$, for all $f \in A(X)$. We will make use of Millman's converse to the Krein-Millman theorem, henceforth referred to as MT, which states that $X = \overline{\text{con}}S$ implies that $X_e \subseteq \overline{S}$. M_x^F will denote the set of representing measures for x which are supported by F i.e. vanish on $X \setminus F$.

1.1. Definition. A compact convex set X is said to have the equal support property (e.s.p) if, for each x in X any two α-maximal representing measures

[1]) I am indebted to Dr. F. Jellett for introducing me to the theory of compact convex sets and for many helpful comments.

for x have the same support. The equal support property was first considered by Feinberg [3] and later by McDonald [2].

A compact convex set X is said to have the strong equal support property (s.e.s.p.) if X_e is closed and X has (e.s.p.). We claim that X has s.e.s.p. if and only if for each $x \in X$, all measures in $M_x^{\overline{X}_e}$ have the same support.

The "only if" part of the previous statement follows from the fact that if supp $\mu \subseteq X_e$, then μ is α-maximal [4, pp.26-27]. Suppose that $z \in \overline{X}_e$ and that all measures in $M_z^{\overline{X}_e}$ have the same support. Let v be α-maximal measure with $\varepsilon_z < v$. Since all α-maximal measures are supported by $\overline{X}_e$ [4, p.30], it follows that $\varepsilon_z = v$. Thus, $z \in X_e$ [4, p.27] ($\varepsilon_z =$ the Dirac measure at z).

2. Some properties of compact convex sets with e.s.p. We show that the class of compact convex sets with e.s.p. does not contain the infinite-dimensional symmetric convex sets as members and consequently are not closed in general under addition and convex combinations, a defect shared with $\alpha(\beta)$-polytopes of Phelps [5].

2.1. Proposition. If X is an infinite dimensional centrally symmetric compact convex set then X cannot have the e.s.p.

Proof. We assume that 0 is a centre of symmetry for X. Since $x \in X_e$ implies that $-x \in X_e$, each of the measures $\mu_x = \frac{1}{2}\varepsilon_x + \frac{1}{2}\varepsilon_{(-x)}$ is maximal and has the origin 0 as harycentre.

Let $y \neq x \in X_e$; then $\mu_y = \frac{1}{2}\varepsilon_y + \varepsilon_{-y}$ is also a maximal measure with barycentre $\underline{0}$. It is clear that μ_y and μ_x have different supports.

Remarks. Above fact makes it easy to show that equal support property is not closed under many of the operations which preserve the finite dimensional polytopes.

For instance, if S is a compact convex set let $K_1 = S$, $K_2 = -S$ then con $(K_1 \cup K_2)$ does not have the equal support property. The same conclusion holds for $K_1 + K_2$. Similarly, although the set $[-1, 1]$ R is a one dimensional simplex, the countable product of it with itself (as a subset of the countable product of lines) is centrally symmetric, hence does not have the e.s.p.

The next example shows that there exist compact convex sets with equal support property such that their intersection does not have e.s.p.

2.2. Example. Let l_1 be the space of absolutely summable real sequence $x = (x_n)_1$, we let

$$S = \{x : x_n \geq 0 \text{ for each } n \text{ and } \Sigma x_n = 1\} .$$

We consider l_1 as the dual of the space c. If all convergent real sequences $y = (y_n)$ with $y_1 = \lim y_n$. Under the w^*-topology defined on l_1 by c, the set S is a compact simplex and if δ_n is the sequence which equals 1 at n, 0 elsewhere then $S_e = \{\delta_n\}$.

Let

$$x = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \ldots\right)$$

so that $x \in S$.

Define $S_1 = S - x$ and let $S_2 = x - S = - S_1$. Both S_1 and S_2 are simplexes and $S_1 \cap S_2$ is centrally symmetric. For $n > 1$ $\frac{1}{2^{n+1}} (\delta_n - \delta_1)$ is in $S_1 \cap S_2$ so $S_1 \cap S_2$ can not have e.s.p. by proposition 2.1.

The following example is essentially due to Phelps [5] and shows that $\alpha(\beta)$-polytopes and compact convex sets with equal support property are distinct generalizations of finite dimensional simplexes.

2.3. Example. Consider the compact convex set S of example 2.2. Define f on S by

$$f(x) = x_2 + x_3 - x_4 - x_5 .$$

Then f is continuous and affine. Therefore $K = S \cap f^{-1}(0)$ is a β-polytope. Furthermore, the map $\phi : S \to K$ defined by

$$\phi(x) = \left(x_1, \frac{1}{2}(x_2 + x_4), \frac{1}{2}(x_3 + x_5), \frac{1}{2}(x_2 + x_5), \frac{1}{2}(x_3 + x_4), x_6, x_7, \ldots\right)$$

is a continuous affine surjection, so K is an α-polytope. We note that

$$e_1 = \frac{1}{2}(\delta_2 + \delta_4), \quad e_2 = \frac{1}{2}(\delta_3 + \delta_5)$$

$$e_3 = \frac{1}{2}(\delta_2 + \delta_5), \quad e_4 = \frac{1}{2}(\delta_3 + \delta_4)$$

are distinct extreme points of K.

$$x = \left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, \ldots\right) = \frac{1}{2}(e_1 + e_2) = \frac{1}{2}(e_3 + e_4).$$

If $\frac{1}{2}\varepsilon_{e_1}+\frac{1}{2}\varepsilon_{e_2}=\mu_1$ and $\mu_2=\frac{1}{2}e_{e_2}+\frac{1}{2}e_{e_4}$ then μ_1 and μ_2 are maximal probability measures with barycentre x and they clearly have distinct supports. Therefore K does not have the e.s.p.

We continue by analysing the concept of a closed face of a compact convex set with e.s.p.

2.4. Definition. Let X be a compact convex set then a convex subset F of X is said to be face if $\lambda x_1+(1-\lambda)x_2 \in F$ implies $x_1, x_2 \in F$ where $x_i \in X\,(i=1,2)$, $0<\lambda<1$.

2.5. Proposition. Let X be a compact convex set with e.s.p. then every closed face of X also has the e.s.p.

Proof. This is immediate from the fact that if F is a closed face of X and if $\mu \in M_1^+(X)$ with barycentre in F, then supp $(\mu) \subseteq F$ and that the maximal measures in $M_1^+(F)$ extend to maximal in $M_1^+(X)$.

If $\{X_\alpha\}$ is an arbitrary family of compact convex sets and if the cartesian product $\Pi_\alpha X_\alpha$ has e.s.p. then each X_α has e.s.p. It is clear from the definition of a face that a face of a face is a face, and so, if F is a face of the convex set X, a point x of X is in F_e if and only if $x \in X_e$. For compact convex sets with the s.e.s.p. we have the following satisfactory situation.

2.6. Proposition. Let X be a compact convex set with s.e.s.p. Then

(*i*) a closed convex subset F of X is a face of X if and only if $F_e \subseteq X_e$.

(*ii*) if F is a face of X then $\overline{F}$ is also a face of X.

Proof. (*i*) follows from [2, theorem 1.8]. Alfen's proof for Bauer simplex is valid for (*ii*) [1, theorem II.7.19].

Remarks. By an example due to Alfsen [6] we know that (*ii*) is inexact for compact convex sets with e.s.p.

Remarks. Among compact convex sets with e.s.p. simplexes are known to satisfy the first part of proposition 2.6. (cf. Jellet) [7]. Although (*i*) can easily be shown to hold for metrizable compact convex sets with e.s.p. we don't know whether this is true for the general case.

It is also not known whether compact convex sets with equal support property (or with s.e.s.p.) have the following properties :

(1) A closed G_δ face is exposed.

(2) A continuous affine function defined a closed face can be extended continuously to the whole convex set.

In the following using a technique first introduced by Asimov [8] we show that every G_δ extreme point in a compact convex set with e.s.p. is in fact exposed.

Let F be a closed face of X and let $[F]$ be the linear subspace (not necessarily closed) spanned by F in $A(X)^*$. Let $A_F(X)$ denote $\{f \in A(X) : f \equiv 0 \text{ on } F\}$. We call F is exposed in the w^*-topology on X if there exists an $f \in A_F(X)$ such that $f(x) > 0$ for all $x \in X \setminus F$.

2.7. Definition. Let $\underline{0}$ be an extreme point of the compact convex set X and let p be the Minkowski functional of X. Let α be a real number and n a positive integer such that $\alpha \geqq 1$ and $n > 1$.

We say that X is (α, n) - additive at $\underline{0}$ if $p(x_1) + \ldots + p(x_n)\, \alpha \leqq p(x_1 + \ldots + x_n)$ for any $x_1, \ldots, x_n \in X$. Call X is α-additive at $\underline{0}$ if X is (α, n)-additive for all n.

We say that X is α-conical at 0 if there exists a (not necessarily continuous) linear functional f on $[X]$ such that $0 \leq f \leq \alpha$ on X and $x \in X$ implies $x \in f(x) X$.

As the following result of Asimov [8] shows, if $\alpha = 1$ then all of the above properties are equivalent.

2.8. Lemma. If X is α-conical at $\underline{0}$ then X is α-additive at $\underline{0}$. If X is α-additive at 0 then X is α^2-conical at 0.

2.9. Lemma. Let X be a compact convex set with $\underline{0}$ as an extreme point and let p be the Minkowski functional of X If $x \in X$ then $p(x) = 1$ if and only if every probability measure on X representing x has mass zero at $\{0\}$.

Proof. Suppose μ_x represents $x \neq 0$ and $\mu(\{0\}) = a > 0$. Then $(\mu_x - a\,\varepsilon_0) / (1 - a)$ is a probability measure on X representing $x / (1 - a)$. Thus $x / (1 - a) \in X$ [1, p. 13] and $p(x) \leqq 1 - a < 1$. Conversely if $p(x) < 1$ then $x = ay\ (a < 1)$.

Let μ be a probability measure representing y. Then $(1-a)\,\varepsilon_0 + a\mu$ represents X and has positive mass at $\{0\}$.

2.10. Proposition. A compact convex set X with e.s.p. is 1-conical at each extreme point.

Proof. We assume $\underline{0}$ is in X_e and show that X is 1-conical at $\underline{0}$. By lemma 2.8 it suffices to show X is 1-additive at $\underline{0}$, or equivalently the set $\{x : p(x) = 1\}$ is convex.

Suppose $z = \lambda x + (1 - \lambda) y$ where $p(x) = 1 = p(y)$ and $0 < \lambda < 1$. Suppose $p(z) = a < 1$ and let $w = \frac{z}{a} \in X$ Let μ_x, μ_y and μ_w be maximal probability measures on X representing x, y, w respectively.

If $\mu_1 = \lambda\mu_x + (1 - \lambda)\mu_y$ and $\mu_2 = (1 - a)\varepsilon_0 + a\mu_w$ then μ_1 and μ_2 are maximal measures representing z. The fact that $\mu_x(\{0\}) = 0$ (lemma 2.9) clearly implies that supp (μ_1) is different from supp (μ_2). A contradiction.

2.11. Corollary. Every G_δ - extreme point x of X is exposed.

Proof. As X is 1-conical at x, it follows from a result of Asimov [8, prop. 2.4] that $A_x(X)$ is approximately 1-directed. The corollary follows by virtue of [8, theorem 2.1].

Remarks. It is well-known that a compact convex set X is a Bauer simplex if and only if for every continuous convex function f on X, the upper envelope, $\hat{f}$ [1, p.II, 4.1] is continuous and affine. In [9], Lima studies compact convex sets with the property that $\hat{f}$ is merely continuous for every continuous convex function, and calls such sets as CE-compact convex sets. We note that using a characterization of compact convex sets with s.e.s.p. as facial quotients of Bauer simplexes [10] it is easy to see that every compact convex set with s.e.s.p. is a CE-compact convex.

3. The Structure Topology. Let Z be a simplex. Effros [11, p. 117] has defined a topology for Z_e called the structure topology. McDonald [2] extended Effros' definition to a larger class of compact convex sets. For the rest of this section $\mathscr{F}(X)$ will denote the collection of closed faces of X.

Let $\mathscr{F}_X = \{F \cap X_e : F \in \mathscr{F}(X)\}$. Note that Φ, $X_e \in \mathscr{F}_X$ and that the intersection of any sub-collection of $\mathscr{F}_X$ is in $\mathscr{F}_X$. Suppose X has the e.s.p. Consider F, $F' \in \mathscr{F}(X)$. By theorem 1.4 of [2] conv $(F \cup F') \in \mathscr{F}(X)$ and it is easy to see that $(F \cap X_e) \cup (F' \cap X_e) = X_e \cap$ conv $(F \cup F')$. It follows that the union of any finite sub-collection of $\mathscr{F}_X$ is in $\mathscr{F}_X$.

Thus, $\mathscr{F}_X$ is the collection of closed sets for a topology on X_e, whenever X has e.s.p.

3.1. Definition. If $\mathscr{F}_X$ is closed under finite unions, then the topology on X_e for which $\mathscr{F}_X$ is the collection of closed sets will be called the structure topology.

3.2. THEOREM. The following are equivalent :

(*i*) The structure topology exists on X_e.

(*ii*) conv $(F \cup F') \in \mathscr{F}(X)$ for all $F, F' \in \mathscr{F}(X)$.

(*iii*) conv $(F \cup F') = F \vee F'$ " " "

(*iv*) $\mathscr{F}(X)$ is a distributive lattice.

Proof. See theorem 3.2. in [2].

Remarks. We note that when the structure topology is defined on X_e, structurally closed sets are of the form $F = T_e$, for some $T \in \mathscr{F}(X)$, but the converse need not hold.

Effros [11] has shown that the structure space need not be locally compact. He conjectures that if X is metrizable then the properties of local compactness, first countability and second countability are equivalent for the structure space of X. In [12] it has been shown that this is indeed the case for Simplexes. For $x \in X$, we take $T(x)$ to be the minimal element of $\mathscr{F}(X)$ which contains x.

We let $\Phi(x) = X_e \cap T(x)$ for each X.

We say that the structure topology satisfies condition (*) if the following holds:

(*) If a sequence $\{x_n\} \subseteq X_e$ converges to x, then all the cluster points of $\{x_n\}$ (with respect to structure topology) lie in $\Phi(x)$.

The following Theorem was given in [13]:

3.3. THEOREM. Suppose X is a metrizable compact convex set satisfying (*). Then the following are equivalent for a fixed $x \in X_e$:

(*i*) The s-topology is first-countable at x.

(*ii*) " " " locally compact at x.

(*iii*) " " " locally sequentially compact at x.

Further, if the s-topology is first countable for each x in X_e, then it is second countable. In the following we show that a compact convex set with equal support property satisfies (*) for the structure topology.

3.4. Proposition. If X is a metrizable compact convex set with e.s.p. then X has property (*).

Proof. Let $\{x_n\}$ be a sequence in X_e converging to x. Suppose $\{x_\beta\}$ is a subnet of $\{x_n\}$ structurally converging to z in X_e. We want to show z is in $T(x)$. Suppose the contrary. Then for some integer N, $n > N, x_n \neq z$. Let

$R = \{x_n : n > N\} \cup T(x)$. Then R is a closed subset of X. Suppose for a moment that the closed convex hull F of R is a face of X. Then z is not an element of F, for otherwise it would be an extreme point of F and hence an element of R by MT. A contradiction. So $X_e \setminus F$ is a neighborhood of z in the structure topology. Since $\{x_\beta\}$ is structurally convergent to z, eventually $\{x_\beta\}$ is not in F. But $\{x_\beta\}$ is a subnet of $\{x_n\}$ and $\{x_n\}$ is eventually in R. A contradiction. It now remains to show $F = \overline{\text{conv}}\ R$ is a face of X Suppose that $p \in F$, and that $p = \lambda p_1 + (1-\lambda) p_2$ where $p_i \in X (i = 1, 2)$ and $0 < \lambda < 1$.

Let v_i be maximal probability measures on X with barycentres p_i respectively. Then $v = \lambda v_1 + (1-\lambda) v_2$ is a maximal measure with barycentre p. It suffices to show supp $(v) \subseteq F$, since then supp $(v_i) \subseteq F$, and as F is closed and convex this would imply that $p_i \in F$. Since R is compact, by integral form of the KM theorem there exists a probability measure μ on X such that supp $(\mu) \subseteq R$ which represents p. We can find a maximal probability measure v_0 representing the point p such that $\mu < v_0$. As X is metrizable, by a Theorem of Cartier [1, p.27] v_0 is a dilation of μ. i.e., there exists a map $X \to \lambda_x$ defined μ-almost everywhere on X with values in $M_1^+(X)$ satisfying

(*i*) $X \to \lambda_x(f)$ is measurable for every $f \in C_{\mathbf{R}}(X)$,

(*ii*) $v_0(f) = \int \lambda_x(f)\, d\mu(x)$ for every $f \in C_{\mathbf{R}}(X)$,

(*iii*) λ_x represents x for μ-almost all x in X.

Consider the closed set R. Let $x \in R$ then x is either in x_n for some $n > N$ or $x \in T(x)$. If $x = x_n$ then $\lambda_x = \varepsilon_{x_n}$ as $x_n \in X_e$ [1, 1.2.4.]. If $x \in T(x)$ then supp $(\lambda_x) \subseteq T(x)$. Therefore supp $(\lambda_x) \subseteq R$ for all $x \in R$.

Suppose $f \in C_{\mathbf{R}}(X)$ such that supp $(f) \cap R = \Phi$ then $\lambda_x(f) = 0$ for all x in R.

Since supp $(\mu) \subseteq R$, we have from (*ii*) that $v_0(f) = 0$. Thus supp $(v_0) \subseteq R$.

As X has e.s.p. it follows that supp $v \subseteq R$. Therefore we can conclude that if X is a metrizable compact convex set with e.s.p. then first countability, local compactness and local sequential compactness are equivalent at a point x of the extreme boundary by virtue of Gleits theorem.

3.5. Corollary. Suppose X is a metrizable compact convex set with e.s.p. and R is a structurally compact subset of X_e. Suppose $a \in A(X)$, $a > 0$ and $a(r) > 0$ for all r in R. Then there exists an $\varepsilon > 0$ such that $a(r) > \varepsilon$ for all r in R.

Proof. If not, then for each positive integer n, choose r_n in R such that $a(r_n) < 1/n$. Choose a subsequence $\{r_m\}$ which converges to y in X. Then $a(y) = \lim a(r_n) = 0$. Choose a subnet $\{r_\beta\}$ of $\{r_m\}$ which structurally converges to some x in R. By the proposition x is in $T(y)$. As $a(y) = 0$ we can easily deduce that a $(T(y)) = 0$. In particular $a(x) = 0$. A contradiction.

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Ö Z E T

Bu çalışmada, eşit destek özeliği denilen bir özeliği taşıyan kompakt, konveks cümleler incelenmektedir.