A NOTE ON DIRECT LIMITS

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In this paper it has been studied hulls of direct limits in the categories of ordered vector spaces and positive linear maps.

Introduction. Let E be a vector space over the reals. A non-empty convex subset K of E is called a wedge if $\lambda K \subseteq K$ for all $\lambda \geq 0$. Clearly a wedge K in E determines a transitive and reflexive relation '$\leq$' by $x \leq y$ if $y - x \in K$; moreover this relation is compatible with the vector structure, i.e.

(*a*) if $x \geq 0$ and $y \geq 0$ then $x + y \geq 0$,

(*b*) if $x \geq 0$ then $\lambda x \geq 0$ for all $\lambda \geq 0$.

The relation determined by a wedge K is called a partial ordering of E, and the pair (E, K) is referred to as a partially ordered vector space.

A wedge K in E is said to be a cone if $K \cap (-K) = \{0\}$. The vector ordering '$\leq$' of E, induced by a wedge C, is antisymmetric if and only if C is a cone.

It is easily seen that the intersection of a family of wedges in E is a wedge. The smallest wedge containing a given set A is denoted by $C(A)$. Clearly

$$C(A) = \left\{ \sum_{i=1}^{n} \lambda_i a_i : a_i \in A, \lambda_i \geq 0 \text{ for all } i = 1,2,\ldots, n \right\}.$$

We see that $C(A)$ is a cone if and only if $\lambda_i = 0$ $(i = 1, 2, \ldots, n)$ whenever $\sum_{i=1}^{n} \lambda_i a_i = 0$, where $a_i \in A \mid \{0\}$ and $\lambda_i \geq 0$ $(i = 1, 2, \ldots, n)$.

If a vector space E is the linear span of certain linear subspaces F_α then we write $E = \sum_\alpha F_\alpha$ $(\alpha \in I)$. Of particular interest to us is the case when each F_α is given as a linear image $T_\alpha(E_\alpha)$ of a linear space E_α. We then write $E = \sum_\alpha T_\alpha(E_\alpha)$.

[1] I would like to thank Dr. F. Jellett for suggesting the present study and for his helpful guidance throughout.

1.1. Definition. Suppose that E_1 and E_2 are ordered vector spaces with cones K_1 and K_2, respectively. A linear mapping T of E_1 into E_2 is said to be positive if $T(K_1) \subset K_2$, that is, $Tx \geq 0$ whenever $x \geq 0$.

If all the E_α $(\alpha \in I)$ in $E = \sum_\alpha T_\alpha (E_\alpha)$ are ordered vector spaces, we can introduce as natural an order as possible on E. The smallest wedge K for which all the T_α's are positive maps from E_α to E suggests itself. Note that the wedge K may not be cone.

1.2. Example. Let N be the subspace of constant functions in $C[0,2]$. Let T be the canonical surjection of $C[0,2]$ onto $C[0,2]/N$. In this case the canonical image of cone in $C[0,2]$ is the smallest wedge for which T is positive, but this image is not a cone. Because the canonical image of the cone coincides with $C[0,2]/N$ and $C[0,2]/N$ can be represented by the space of all real-valued, continuous functions on $[0,2]$ that vanish at 0.

1.3. Definition. If the smallest wedge K in $E = \sum_\alpha T_\alpha (E_\alpha)$ for which all the T_α are positive maps, is a cone then E is called the order-hull of the spaces (E_α, K_α) by the maps T_α $(\alpha \in I)$.

2. If $\{E_\alpha : \alpha \in I\}$ is an arbitrary family of ordered vector spaces and K_α is the positive cone in E_α for each $\alpha \in I$, then the direct sum $\bigoplus_{\alpha \in I} E_\alpha$ is an ordered vector space for the cone $K = \bigoplus_{\alpha \in I} K_\alpha$.

2.1. Every order-hull E of ordered vector spaces is order isomorphic to a quotient of the ordered direct sum by an order ideal.

Proof. Consider the map $T\colon \bigoplus_\alpha E_\alpha$ onto E defined by $T(\Sigma x_\alpha) = \Sigma T_\alpha x_\alpha$. If H is the kernel of T, then E is order isomorphic to $\oplus E_\alpha / H$. We also have the converse.

2.2. Every quotient of the ordered direct sum by an order ideal is order isomorphic to an order-hull. We note that the wedge K in $\sum_\alpha T_\alpha (E_\alpha)$ is a cone if and only if the null space H of the mapping T which establishes the order isomorphism 2.1 is an order ideal.

It is easy to see that a linear mapping T from an order-hull $E = \sum_\alpha T_\alpha (E_\alpha)$ into an ordered linear space F is positive if and only if all the maps TT_α $(\alpha \in I)$ are positive.

The requirements for the construction of an order-hull are often realized in the following special form : The index set I is directed and for each α, β in I, $\alpha \leq \beta$ there exists a positive linear map $T_{\alpha\beta}$ from $T_\alpha(E_\alpha)$ into $T_\beta(E_\beta)$ such that

$$T_\alpha = T_\beta \, T_{\alpha\beta} \quad \text{whenever} \quad \alpha \leq \beta. \qquad *1.$$

We note that if the maps T_α, $\alpha \in I$, in *1 are one-one, then so are the maps $T_{\alpha\beta}$, and we have

$$T_{\gamma\beta} \, T_{\beta\alpha} = T_{\gamma\alpha} \text{ for } \alpha \leq \beta \leq \gamma. \qquad *2.$$

With trivial modifications of Köthe's arguments in [1, p.9,218], one can show that every order-hull can be expressed in above form.

Remark. Suppose (E, K) $(= \sum_\alpha T_\alpha E_\alpha)$ and that the maps T_α are one-one, and there exist maps $T_{\alpha\beta}$ for which *1 and *2 are satisfied then K is a cone in E.

2.3. If $E (= \sum_\alpha T_\alpha E_\alpha)$ is the order-hull of the ordered linear spaces E_α and if mappings $T_{\alpha\beta}$ are given which satisfy *1, then the quotient space $E' = \Sigma \phi_\alpha(E_\alpha)$ isomorphic to E satisfies the following relations :

$$\phi_\alpha = \phi_\beta \, T_{\alpha\beta} \text{ for } \alpha < \beta,$$

where ϕ_α is the restriction of the quotient map to E_α.

So far we have started from a vector space E together with subspaces $T_\alpha(E_\alpha)$ and morphisms $T_{\alpha\beta}$. It is natural to ask how far E is determined by the E_α and the $T_{\alpha\beta}$, and whether given E_α and $T_{\alpha\beta}$, we can give an ordered vector space E and positive linear maps T_α which satisfy *1.

Suppose $(E_\alpha, K_\alpha, \alpha \in I)$ be a family in the category of ordered vector spaces and positive linear maps. $\{E_\alpha, K_\alpha\}$ is said to be a direct family if it satisfies the following :

For each $\alpha, \beta \in I$, $\alpha \leq \beta$, there exists a positive linear map $T_{\alpha\beta} : E_\alpha \to E_\beta$ such that

a. $T_{\alpha\alpha} = i_{E_\alpha}$

b. $T_\alpha = T_\beta \, T_{\alpha\beta}$.

A direct limit of this family is an object E, together with a family of morphisms $T_\alpha : E_\alpha \to E$ for each $\alpha \in I$, such that $T_\beta \, T_{\alpha\beta} = T_\alpha$ whenever $\alpha \leq \beta$. Moreover, if F is another object with morphisms $G_\alpha : E_\alpha \to F$ with $G_\beta \, T_{\alpha\beta} = G_\alpha$ whenever $\alpha \leq \beta$, then there exists a unique morphism T of E into F such that $T \, T_\alpha = G_\alpha$ for each $\alpha \in I$.

It is clear that if a direct limit exists then it is unique upto a commuting isomorphism.

We now show that direct limits exist in the category of ordered linear spaces and positive linear maps and give a representation :

2.4. Proposition. Let E_0 be the ordered direct sum of the E_α's and H be the subspace of E_0 generated by the ranges of the linear maps $J_\alpha \rightarrow J_\beta T_{\alpha\beta}$ of E_α into E_β, where (α, β) runs through all pairs such that α β. Then $(E_0/H, \phi_\alpha)$, where ϕ_α is the restriction of the canonical map $\phi : E_0 \rightarrow E_0/H$ to E_α, for each α, is the direct limit.

Proof. An element $x = \sum_{i=1}^{n} x_{\alpha_i}$ of E_0 lies in H if and only if there exists an index $\beta \geq \alpha_i$ $(i = 1, 2, \ldots, n)$ such that $\Sigma T_{\alpha_i\beta} x_{\alpha_i} = 0$ [1, p.9, 219]. Using this characterization of H, it is easy to see that H is an order ideal, so that the quotient order is well defined. The rest is straightforward.

Remark. Even when all the maps $T_{\alpha\beta}$ are non-zero it can happen that the direct limit is 0 alone. A sufficient condition for the limit to be non-zero is that all the $T_{\alpha\beta}$ are one-one [1, p.9, 219]. The following gives conditions for an order-hull to be representable as a limit.

2.5. If $E(= \sum_{\alpha} T_\alpha E_\alpha)$ is an-order hull, and if $T_{\alpha\beta}$ is a system of morphisms satisfying *1 and *2, then E is a homomorphic image of the direct limit of the E_α.

A necessary and sufficient condition for E to be isomorphic to the limit is that an equation $T_\alpha x_\alpha = 0$ $(x_\alpha \neq 0)$ holds in E if and only if there exists $\beta > \alpha$ (depending on x_o) such that $T_{\alpha\beta} x_\alpha = 0$. As direct limits are shown to be quotient of direct sums, the following order-theoretical properties of the limits follow easily from those of the objects in the family.

2.6. Definition. Suppose that K is the positive cone in an ordered linear space E. K generates E if E is the linear subspace spanned by K, that is $E = K - K$. E is almost Archimedean if $-\alpha y \leq x \leq \alpha y$ for some $y \in K$, and all real numbers $\alpha > 0$ implies $x = 0$. E is Archimedean if $x \leq 0$ whenever $\alpha x \leq b$ for some $b \in K$ and all $\alpha > 0$. Clearly, every Archimedean ordered vector space is almost Archimedean.

E is said to have the Riesz decomposition property if $[0, u]+[0, v]=[0, u+v]$ whenever u, v are elements in K.

2.7. Proposition. If each E_α , K_α is positively generated (has *R.D.P.*) then the limit is also positively generated (*R.D.P.*). We note that although the direct sum of a family of Archimedean ordered linear spaces is Archimedean ordered, the direct limit of such spaces need not be Archimedean ordered [2, p.9, 67].

We now introduce the concept of 'positive' partition of unity for an order-hull.

2.8. Definition. Let $E\ (=\sum_\alpha T_\alpha E_\alpha)$ be an order-hull. A set of linear maps $(G_\alpha), G_\alpha : E \to E_\alpha$ for each α, is said to be a partition of unity for E if

(1) G_α is positive for each α,

(2) for each β, $G_\alpha T_\beta = 0$ except for a finite number of indices α,

(3) $\sum_\alpha G_\alpha T_\alpha$ is the identity on E.

Example. The ordered direct sum $\oplus E_\alpha$ of ordered linear spaces E_α has a partition of unity, if one takes G_α to be the canonical projection of $\oplus E_\alpha$ to E_α.

We note that while the order-hull of vector lattices need not be a vector lattice, the existence of a partition of unity seems to be sufficient for preservation of this property.

2.9. Proposition. Let $E\ (=\sum_\alpha T_\alpha E_\alpha)$ be an order-hull with a partition of unity (G_α) then if:

(1) each E_α is Archimedean ordered then so is E,

(2) " " is a vector lattice " " " ",

(3) " " is an order complete vector lattice (σ-order complete vector lattice) then so is E.

Proof. Straightforward.

2.10. Proposition. Let $E(=\sum T_\alpha E_\alpha)$ be an order-hull with a partition of unity (G_α). A subset B of E is order-bounded if and only if it is equal to a finite sum of subsets of E with the same property.

Proof. As $B = \sum_\alpha T_\alpha G_\alpha(B)$, it suffices to show $G_\alpha(B) = 0$ for all but a finite number of indices α_i, $i = 1,2,\dots,n$. Suppose a and c are elements in E such that $a \leq b \leq c$ for all b in B. Suppose $G_\alpha(B) \neq 0$ for all but a finite number of indices. Then there is a sequence $(\alpha(n))$ of indices and a sequence $b_n (\neq 0)$ of elements of B such that $G_{\alpha_n}(b_n) \neq 0$. We can find an index $\alpha(m)$ such that

$G_{\alpha_m}(a) = G_{\alpha_m}(b)=0$. As $G_{\alpha_m}(a) \leq G_{\alpha_m}(b_m) \leq G_{\alpha_m}(c)$, it follows that $G_{\alpha_m}(b_m)=0$. But this contradicts the choice of b_m. Hence $G_\alpha(B) = 0$ for all but finite α.

Observation. If the set of indices is directed and there exist positive linear maps $T_{\alpha\beta}$ from E_α into E_β which commute with T_α whenever $\alpha \leq \beta$ then :

A set B is order bounded in E only if there exists a positive linear map T^* from E into some E_α such that $B = T_\alpha T^*(B)$.

Proof. Let V be the finite set of indices α such that $G_\alpha(B) = 0$ if $\alpha \notin V$. Choose $\beta \in I$ such that $\beta \geq \alpha$, $\alpha \in V$. Then $T^* = \Sigma T_{\alpha_i\beta} G_{\alpha_i}$ is a positive linear map of E into E_β. Moreover we have the following :

$$B = \Sigma T_\alpha G_\alpha(B) = \sum_i T_{\alpha_i} G_{\alpha_i}(B) = T_\beta (\sum_i T_{\alpha_i\beta} G_{\alpha_i}(B)) = T_\beta T^*(B).$$

3. When an ordered linear space is also a topological vector space, the resulting mixed structure is called an ordered topological vector space.

In the following we shall consider hulls of ordered locally convex spaces.

Let $E(=\sum_\alpha T_\alpha E_\alpha)$ be the order-hull of the ordered vector spaces E_α by the maps T_α. If the E_α's are locally convex (Hausdorff) spaces, their topologies can be pieced together to induce a topology on E. The finest locally convex topology τ on E for which all the T_α's are continuous mappings from E_α to E suggests itself.

Note that an absolutely convex subset U of E is a τ-neighborhood of 0 if $T_\alpha^{-1}(U)$ is a neighborhood of 0 in E_α for each α. Thus absolutely convex envelopes of the sets of the form $T_\alpha(U)$ where each U_α is taken to be a neighborhood of 0 in E_α form a base of τ-neighborhoods of 0. This topology need not be Hausdorff, but if this is the case, $E(\tau)$ is called the locally convex order-hull of the E_α's by the maps T_α. The topology τ is called the hull topology on E.

We can say more about the order isomorphism of 2.1.

3.1. Every locally convex order-hull $E(\tau)$ $(= \Sigma T_\alpha (E_\alpha(\tau_\alpha)))$ is topologically order isomorphic to a quotient of the locally convex sum by a closed order ideal.

We note that the assertion corresponding to 2.2 is also true.

The existence of a partition of unity with the additional restriction that the maps (G_α) that form the partition of unity are continuous ensures that the hull topology is Hausdorff. The idea of continuous partition of unity is due to De Wilde [3] and they provide a simple system of semi-norms characterizing the hull topology of E. Let P_α be a system of seminorms defining the topology τ_α of E_α.

3.2. Proposition. Let $E\,(=\Sigma\, T_\alpha\,(E_\alpha\,(\tau_\alpha)))$ be a locally convex order-hull of ordered locally convex spaces with a partition of unity (G_α). Then the hull topology is characterized by the seminorms

$$p(e) = \Sigma\, c_\alpha\, p_\alpha\,(G_\alpha(e)) \text{ where } c_\alpha \geq 0,\ p_\alpha \in P_\alpha .$$

Proof. cf. [3, p.9.2].

3.3. Corollary. The convex hull of separated convex spaces with a continuous partition of unity is separated.

Remarks. Since the definition of an ordered topological vector space does not require any direct relation to exist between the order and topological structures involved, it is necessary to impose further restrictions on the spaces under consideration to obtain a significant theory. In the theory of ordered topological vector spaces two properties play an important role : one condition is to say that the cone is 'large' enough to give an open decomposition property and the other is to say that the cone is 'small' enough, so that the topology admits a neighborhood base at $\underline{0}$ consisting of order-convex sets.

3.4. Definition. Let (E, K, τ) be an ordered topological vector space. If each $V\cap K - V\cap K$ is a τ-neighborhood of 0 whenever V is a τ-neighbourhood of 0 then E is said to have an open decomposition property. Thus K gives an open decomposition in (E, τ) if and only if (E, τ) admits a neighborhood base at $\underline{0}$ consisting of positively generated τ-neighborhoods. Let us call an ordered convex space with the decomposition property locally decomposable.

3.5. Proposition. Let $E\ (=\Sigma\, T_\alpha\, E_\alpha\,(\tau_\alpha))$ be an ordered convex hull of locally decomposable spaces, then the hull topology is locally decomposable.

Proof. A neighborhood base at 0 for the hull topology is provided by the family of all sets of the form $V = \text{conv}\,({}_{\alpha\in I}\, T_\alpha\,(V_\alpha))$ where each V_α is a τ_α basic neighborhood of 0 in E_α. Since τ_α is locally decomposable, we can take V_α to be circled, convex and decomposable. Since T_α is positive, $T_\alpha(V_\circ)$ must also be decomposable. Consequently V, as the convex hull of decomposable sets, must be decomposable.

3.6. Corollary. Direct sums, quotients and direct limit of decomposable spaces are decomposable.

3.7. Definition. Let (E, K) be an ordered vector space. A subset A of E is order convex if $[a_1, a_2] \subseteq A$ whenever $a_1, a_2 \in A$ and $a_1 \leq a_2$. A vector topology τ in (E, K) is said to be locally order convex if it admits a neighborhood base at 0 consisting of order convex sets. In this case, we say that (E, K, τ) is a locally order convex space and the cone K is called a normal cone in (E, τ).

We note that in general normality is not preserved under taking order-hulls, but as we will see, the existence of a partition of unity is a sufficient condition for the permanence of this property.

3.8. Proposition. Let $E(=\Sigma\, T_\alpha\, E_\alpha\,(\tau_\alpha))$ be the ordered convex hull of locally ordered convex spaces $(E_\alpha\,,\ \tau_\alpha)$ with a continuous partition of unity, then the hull topology is locally order convex.

Proof. It is known [4, prop. 1.5, p. 63] that an ordered convex space is locally order convex if there exists a family of seminorms (p_α) generating the topology such that $p_\alpha(z) \geq p_\alpha(y)$, for each α, whenever $z \geq y$. Using 3.2 it is straightforward to show that the defining family of seminorms for the hull topology are monotone.

3.9. A direct sum of locally order convex spaces is locally order convex.

4. In this section we show that many objects, in the category of ordered topological vector spaces and positive linear maps, can be written as order hulls of basic building blocks such as the order-unit spaces, etc.

4.1. Definition. A subset H of the positive cone K in an ordered vector space E exhausts K if, for each $x \in K$, there are $h \in H$ and $\lambda > 0$ such that $x \leq \lambda h$. An exhausting set consisting of a single point e is said to be an order unit for E. If E is an ordered vector space, the order topology τ_0 on E is the finest locally convex topology τ on E for which every order bounded set is τ-bounded.

The next result shows that the order topology is a hull topology.

4.2. Proposition. Suppose that E is an almost Archimedean ordered vector space and that H is a subset of the positive core that exhausts it. For each $h \in H$, h is an order unit in the linear subspace $E_h = \bigcup_n n[-h, h]$ of E and the order topology $\tau_0^{(h)}$ on E_h is generated by a norm. Then (E, τ_0) is the ordered convex hull of the subspaces $(E_h\,(\tau_0^{(h)}) : h \in H)$.

Proof. cf. [4, prop. 1.11, p.122].

Remark. It is known [4, p. 125] that the order topology agrees with the given topology in any complete, metrizable, locally convex lattice, thereby providing a decsription of this wide class of spaces as hulls.

4.3. Definitions. A subset A of a vector lattice E is solid if $y \in A$ whenever $x \in A$ and $|y| \leq |x|$. An ordered topological vector space $E(\tau)$ which is a lattice is a topological vector lattice if there is a neighborhood basis of 0 for τ consisting of solid sets. In addition, if $E(\tau)$ is a locally convex space, then $E(\tau)$ is a locally convex lattice. A vector lattice E equipped with a norm 11.11 is a normed vector lattice if $|x| < |y|$ implies $\|x\| \leq \|y\|$.

Let us call a linear map between two vector lattices an 1-homomorphism if it preserves the lattice operations. The next result is due to Kawai [5].

4.4. Proposition. Every bornological vector lattice (E, K, τ) is the topological order-hull of a family of normed vector lattices (and of Banach lattices if E is quasi-complete for τ) with respect to lattice homomorphism. The cardinality of this family can be chosen to be the cardinality of any fundamental system of τ-bounded sets in E.

The last result of the present section is due to us. For a proof we refer to [6].

4.5. Proposition. Let $(E, K, 11.11)$ be an almost Archimedean ordered Banach space with a closed generating, normal cone. Then E is the topological order-hull of a family of $A(K_\alpha)$ spaces for some compact convex sets K_α.

5. Further Remarks. When a vector space is equipped with a lattice ordering certain notions of convergence can be defined in terms of the order structure. Moreover, each type of convergence determines a corresponding continuity concept for linear maps. A net (y_α) in a vector lattice E **decreases** to y_0 if $y_0 = \inf (y_\alpha)$ and $y_\alpha \geq y_\beta$ whenever $\alpha \geq \beta$. A net (x_α) **order converges** to x_0, if (x_α) is an order bounded subset of E and there exists a net (y_α) that decreases to 0 such that $|x_\alpha - x_0| \leq y_\alpha$ for all α. Consequently, if T is a linear mapping of E into F then T is order continuous if the net (Tx_α) order converges to 0 in F whenever (x_α) is a net that order converges to $\underline{0}$ in E. Finally a vector lattice E is said to have the diagonal property if, whenever $(x^{(n,m)} : n,m = 1,2,\ldots) \subseteq E$ and

a. $(x^{(n,m)})$ order converges to $x^n \in E$ for each n,

b. $(x^{(n)})$ order converges to $x^{(0)} \in E$,

then there exists a strictly increasing sequence $(m_n : n = 1,2,\ldots)$ of positive integers such that $(x^{(n,m_n)})$ order converges to $x^{(0)}$.

Observation. If $E = \Sigma\, T_\alpha (E_\alpha)$ is an order-hull of o-order complete Archimedean vector lattice with diagonal property then each T_α is sequentially order continuous if E has a partition of unity. This is an easy consequence of [4, prop. 5.16] and proposition 2.9.

An ordered vector space E is order separable if every subset A of E that has a supremum in E contains a countable subset A' such that $\sup A = \sup A'$. The restriction in this definition plays a major role in establishing the relation between various types of continuity mentioned above. Namely, every positive sequentially order continuous linear map T of an order separable vector lattice E into an Archimedean vector lattice F is order continuous. Hence we can conclude : if $E = \sum_{\alpha} T_{\alpha} E_{\alpha}$ is an order-hull of σ-order complete, order separable, Archimedean vector lattices with the diagonal property, then each T_{α} is order continuous, if E has a partition of unity.

Let E_1, E_2 be ordered vector spaces and let $T: E_1 \to E_2$ be a positive linear operator. Jellett [7] calls T an R-homomorphism if the following condition is satisfied:

For every pair $a,b \in E_1$ and $d \in E_2$ such that $Ta,\ Tb \leq d$, there exists $c \in E_1$ such that $a,b \leq c$ and $Tc \leq d$.

It is easy to see that ordered vector spaces with the decomposition property and R-homomorphism form a category. It is not known that direct limits exist in this category.

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ÖZET

Bu çalışmada, sıralanmış vektör uzayları ve pozitif lineer tasvirler kategorilerinde direkt limitlerin kapanışları incelenmektedir.