PROJECTIVE AFFINE MOTION IN A PRFn-SPACE, V

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In this paper has been investigated in a PRFn-space the existence of projective affine motion characterized by

$$\bar{x}^i = x^i + v^i(x)\,dt$$

and $\pounds_u G^i{}_{jk} = 0$ of several types.

1. INTRODUCTION

Let Fn [1][1)] be an n-dimensional affinely connected and non-flat Finsler space equipped with symmetric Berwald's connection coefficient $G^i_{jk}(x,\dot{x})$. The covariant derivative of any tensor field $T^i_j(x,\dot{x})$ with respect to x^k in the sense of Berwald is given by

$$T^i_{j(k)} = \partial_k T^i_j - \dot{\partial}_m T^i_j G^m_{\gamma k}\dot{x}^\gamma + T^s_j G^i_{sk} - T^i_s G^s_{jk}. \tag{1.1}$$

The commutation formula involving the Berwald's covariant derivative is given by [1]:

$$2\,T^i_{j[(h)(k)]} = -\dot{\partial}_\gamma T^i_j H^\gamma_{shk}\dot{x}^s + T^s_j H^i_{shk} - T^i_s H^s_{jhk}, \text{}^{2),\,3)} \tag{1.2}$$

where

$$H^i_{hjk}(x,\dot{x}) \overset{\text{def.}}{=} 2\,\{\partial_{[k} G^i_{j]h} - G^i_{\gamma h[j} G^\gamma_{k]s}\dot{x}^s + G^\gamma_{h[j} G^i_{k]\gamma}\} \tag{1.3}$$

is called Berwald's curvature tensor and satisfies the following identities [1]:

$$H^i_{hjk} = -H^i_{hkj}, \tag{1.4}$$

$$H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0 \tag{1.5}$$

and

$$H^i_{hji} = H_{hj}. \tag{1.6}$$

Let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\,dt, \tag{1.7}$$

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1) Numbers in brackets refer to the references at the end of the paper.

2) $2\,A_{[hk]} = A_{hk} - A_{kh}$.

3) $\dot{\partial}_i \equiv \partial/\partial\dot{x}^i$ and $\partial_i \equiv \partial/\partial x^i$.

where $v^i(x)$ is any vector field and dt is an infinitesimal point constant. In view of the above point transformation and Berwald's covariant derivative, the Lie derivatives of $T_j^i(x, \dot{x})$ and $G^i_{jk}(x, \dot{x})$ respectively are given by [2]:

$$\pounds_v T_j^i = T_{j\,(h)}^i v^h - T_j^h v^i_{(h)} + T_h^i v^h_{(j)} + \dot{\partial}_h T_j^i v^h_{(s)} \dot{x}^s \tag{1.8}$$

and

$$\pounds_v G^i_{jk} = v^i_{(j)(k)} - H^i_{jkh} v^h + G^i_{sjk} v^s_{(\gamma)} \dot{x}^\gamma , \tag{1.9}$$

where $G^i_{sjk} \equiv \dot{\partial}_s G^i_{jk}$.

In an Fn if the Berwald's curvature tensor field $H^i_{hjk}(x, \dot{x})$ satisfies the relation

$$H^i_{hjk\,(s)} = \lambda_s H^i_{hjk} , \tag{1.10}$$

where $\lambda_s(x)$ is any covariant vector then the space is called projective recurrent Finsler space of first order or PRFn-space. The present author has investigated in an PRFn-space the existence of projective affine motion characterized by (1.7) and $\pounds_v G^i_{jk} = 0$ of the following several types [4], [5], [6]:

(A) Contra-form characterized by $v^i_{(j)} = 0$,

(B) Concurrent form defined by $v^i_{(j)} = a\,\delta_j^i$ ($a =$ const.),

(C) Special concircular-form introduced by $v^i_{(j)} = \psi(x)\,\delta_j^i$,

(D) Projective recurrent-form characterized by $v^i_{(j)} = \psi_j(x)\, v^i$,

(E) Concircular form satisfying the condition: $v^i_{(j)} = \sigma(x)\,\delta_j^i + \psi_j(x)\, v^i$ ($\psi_j =$ gradient vector).

However, these types are contained, as a special case respectively in the condition

$$v^i_{(j)} = \sigma(x)\,\delta_j^i + \psi_j(x)\, v^i, \tag{1.11}$$

where $\sigma(x)$ means any function and ψ_j denotes a certain covariant vector. The vector field (v^i) defined by such a condition will be called a torse-forming field. Then, as the most general case, in this paper, the author will discuss about the projective affine motion of torse-forming form defined by

$$\overline{x}^i = x^i + v^i(x)\,dt , \quad v^i_{(j)} = \sigma(x)\,\delta_j^i + \psi_j(x)\, v^i . \tag{1.12}$$

2. PROJECTIVE AFFINE MOTION AND THREE CASES

In what follows we shall assume the existence of projective affine motions of torse-forming form (1.12). Then, we come to assume two conditions:

$$\text{a)}\quad \pounds_v\, G^i_{jk} = 0 \quad \text{or} \quad \text{b)}\quad v^i_{(j)(k)} = H^i_{jkh}\, v^h - G^i_{sjk}\, v^s_{(\gamma)}\, \dot{x}^\gamma \tag{2.1}$$

that is the equation of projective affine motion and its integrability condition

$$\pounds_v\, H^i_{hjk} = v^s \lambda_s\, H^i_{hjk} - H^s_{hjk}\, v^i_{(s)} + H^i_{sjk}\, v^s_{(h)} + H^i_{hsk}\, v^s_{(j)} + \\ + H^i_{hjs}\, v^s_{(k)} + \dot{\partial}_s\, H^i_{hjk}\, v^s_{(\gamma)}\, \dot{x}^\gamma = 0. \tag{2.2}$$

From the latter part of (1.12), we can construct

$$v^i_{(j)(k)} = \sigma_k\, \delta_j^i + \psi_{j\,(k)}\, v^i + \psi_j (\sigma\, \delta_k^i + \psi_k\, v^i)\,, \tag{2.3}$$

where

$$\sigma_k = \sigma_{(k)} = \partial_k\, \sigma\,. \tag{2.4}$$

Introducing the latter part of (1.2) and (2.3) into the equation (2.1), we have

$$H^i_{jkh}\, v^h = \psi_{j\,(k)}\, v^i + \psi_j\, \psi_k\, v^i + \sigma_k\, \delta_j^i + \sigma\, \psi_j\, \delta_k^i. \tag{2.5}$$

Transvecting the last formula by v^k and using the fact that $H^i_{jkh}\, v^k\, v^h = 0$, we get

$$\psi_{j\,(k)}\, v^i\, v^k + \psi_j\, \psi_k\, v^i\, v^k + \sigma_k\, \delta_j^i\, v^k + \sigma\, \psi_j\, v^i = 0. \tag{2.6}$$

For the non-zero property of the vector $v^i(x)$ the last relation reduces to

$$\psi_{j\,(k)}\, v^k + \psi_j\, \psi_k\, v^k + \sigma\, \psi_j = 0, \tag{2.7}$$

where we have used the relation [6] :

$$\pounds_v\, \sigma\,(x) = 0 \quad \text{or} \quad \sigma_h\, v^h = 0. \tag{2.8}$$

Equating the indices i and k of the formula (2.5), we obtain

$$H^i_{jih}\, v^h = \psi_{j\,(k)}\, v^k + \psi_j\, \psi_k\, v^k + \sigma_j + n\sigma\, \psi_j\,. \tag{2.9}$$

Introducing (2.7) into the above relation, we find

$$H^i_{jih}\, v^h = (n-1)\, \sigma\, \psi_j + \sigma_j\,. \tag{2.10}$$

Differentiating covariantly the above formula with respect to x^s and taking care of the equations (1.10) and the latter part of (1.12), we have

$$(\psi_s + \lambda_s)\, H^i_{jih}\, v^h + \sigma\, H^i_{jis} = (n-1)\,(\sigma_s\, \psi_j + \sigma\, \psi_{j\,(s)}) + \sigma_{j\,(s)}\,. \tag{2.11}$$

Substituting (2.10) into the left-hand side of the above equation, we get

$$(\psi_s + \lambda_s)\, \{(n-1)\, \sigma\, \psi_j + \sigma_j\} + \sigma\, H^i_{jis} = (n-1)\,(\sigma_s\, \psi_j + \sigma\, \psi_{j\,(s)}) + \sigma_{j\,(s)}. \tag{2.12}$$

Transvecting the above formula by v^s and noting the equations (2.7), (2.8) and (2.10), we obtain

$$(n-1)\,\sigma\,\psi_j\,(2\,\psi_s\,v^s + \lambda_s\,v^s + 2\sigma) + \sigma_j\,(\psi_s\,v^s + \lambda_s\,v^s + 2\sigma) = 0. \tag{2.13}$$

Now, again transvecting the formula (2.13) by v^j and using the equation (2.8), we find

$$(n-1)\,\sigma\,\psi_j\,v^j\,(2\,\psi_s\,v^s + \lambda_s\,v^s + 2\sigma) = 0. \tag{2.14}$$

Consequently, it is seen from (2.14) that in order to discuss the possibility of torse-forming projective affine motion in an PRFn-space it is necessary to investigate the three cases for $n \geqslant 2$:

$$\text{a)}\quad \sigma = 0, \qquad \text{b)}\quad \psi_j\,v^j = 0 \quad \text{and} \quad \text{c)}\quad 2\,\psi_s\,v^s + \lambda_s\,v^s + 2\sigma = 0. \tag{2.15}$$

3. THE CASE OF $\sigma = 0$

In view of this case the projective affine motion (1.12) under consideration is degenerated into

$$\overline{x}^i = x^i + v^i\,(x)\,dt,\ \ v^i_{(j)} = \psi_j\,(x)\,v^i. \tag{3.1}$$

But the above case has been already discussed deeply by the author in the papers [5] and [7] .This is projective recurrent affine motion form.

4. THE CASE OF $\psi_s\,v^s = 0$

Differentiating covariantly (2.5) with respect to x^m and taking notice of the equations (1.10) and the latter part of (1.12), we have

$$\begin{aligned}(\lambda_m + \psi_m)\,H^i{}_{jkh}\,v^h + \sigma\,H^i{}_{jkm} &= \psi_{j\,(k)\,(m)}\,v^i + \psi_{j\,(k)}\,v^i{}_{(m)} + \psi_{j\,(m)}\,\psi_k\,v^i + \\ &+ \psi_j\,\psi_{k\,(m)}\,v^i + \psi_j\,\psi_k\,v^i_{(m)} + \sigma\,\psi_{j\,(m)}\,\delta_k^i + \sigma_m\,\psi_j\,\delta_k^i + \sigma_{k\,(m)}\,\delta_j^i\,.\end{aligned} \tag{4.1}$$

Contracting the above formula with respect to the indices i and j we get

$$\begin{aligned}(\lambda_m + \psi_m)\,H^i{}_{ikh}\,v^h + \sigma\,H^i{}_{ikm} &= \psi_{i\,(k)\,(m)}\,v^i + \psi_{i\,(k)}\,v^i{}_{(m)} + \psi_{i\,(m)}\,\psi_k\,v^i + \\ &+ \psi_i\,\psi_k\,v^i{}_{(m)} + \sigma\,\psi_{k\,(m)} + \sigma_m\,\psi_k + n\,\sigma_{k\,(m)},\end{aligned} \tag{4.2}$$

where we have used the fact that

$$\psi_h\,v^h = 0. \tag{4.3}$$

With the help of the above formula we can deduce

$$\psi_{h\,(s)}\,v^h + \psi_h\,v^h{}_{(s)} = 0. \tag{4.4}$$

Again differentiating covariantly the last formula with respect to x^m and using the latter part of (1.12) we find

$$\psi_{h\,(s)\,(m)}\, v^h + \psi_{h\,(s)}\, v^h{}_{(m)} + \psi_{h\,(m)}\, v^h{}_{(s)} + \psi_h\, (\sigma\, \delta_s^h + \psi_h\, v^h)_{(m)} = 0 \tag{4.5}$$

or

$$\psi_{h\,(s)\,(m)}\, v^h + \psi_{h\,(s)}\, v^h{}_{(m)} + \psi_{h\,(m)}\, \psi_s\, v^h + \sigma\, \psi_{s\,(m)} + \sigma_m\, \psi_s + \psi_s\, \psi_h\, v^h{}_{(m)} = 0, \tag{4.6}$$

where we have used (2.4) and (4.3).

Now, introducing the equation (4.6) into the right-hand side of the formula (4.2), we obtain

$$(\lambda_m + \psi_m)\, H^i{}_{ikh}\, v^h + \sigma H^i{}_{ikm} = n\, \sigma_{k\,(m)}. \tag{4.7}$$

Commutating the last relation with respect to the indices k and m and using the fact that $\sigma_{k(m)} = \sigma_{m(k)}$, we have

$$(\lambda_m + \psi_m)\, H^i{}_{ikh}\, v^h + \sigma\, H^i{}_{ikm} = n\, \sigma_{k\,(m)}. \tag{4.8}$$

The above formula can also be re-written as

$$2\,\sigma\, H^i{}_{ikm} = (\lambda_k + \psi_k)\, H^i{}_{imh}\, v^h - (\lambda_m + \psi_m)\, H^i{}_{ikh}\, v^h. \tag{4.9}$$

Transvecting the last result by v^m and using the fact that $H^i{}_{ikm}\, v^k\, v^m = 0$, we get

$$(2\sigma + \lambda_m\, v^m)\, H^i{}_{ikh}\, v^h = 0, \tag{4.10}$$

where we have used (4.3).

Hence, we have two cases to be discussed:

$$H^i{}_{ikh}\, v^h = 0 \tag{4.11}$$

and

$$2\sigma + \lambda_m\, v^m = 0. \tag{4.12}$$

PROJECTIVE AFFINE MOTION GIVEN BY (4.11)

In such a case, from (2.1), we have

$$v^i{}_{(i)\,(k)} = 0. \tag{4.13}$$

Thus, by virtue of the equations (2.3) and (4.13), we can get

$$n\, \sigma_k + \psi_{h\,(k)}\, v^h + \sigma\, \psi_k = 0. \tag{4.14}$$

In view of the equations (1.12) and (4.3), the last formula takes the form

$$n\, \sigma_k - \psi_h\, (\sigma\, \delta_k^h + \psi_k\, v^h) + \sigma\, \psi_k = 0 \tag{4.15}$$

or

$$\sigma_k = 0. \tag{4.16}$$

Then, we shall have proved $\psi_j = 0$. In what follows we shall always use (4.3) and (4.16). From the formula (2.3), we can get

$$v^i_{(j)(i)} = \sigma_j + \psi_{j(i)} v^i + n \sigma \psi_j . \tag{4.17}$$

By virtue of (4.3), the equation (2.7) takes the form

$$\sigma \psi_j = - \psi_{j(k)} v^k . \tag{4.18}$$

Introducing the last result into the right-hand side of the formula (4.17), we obtain

$$v^i_{(j)(i)} = (n-1) \sigma \psi_j + \sigma_j . \tag{4.19}$$

But, with the help of the equation (4.16), the last relation reduces to

$$v^i_{(j)(i)} = (n-1) \sigma \psi_j . \tag{4.20}$$

From the equations (2.1) and (4.11), we can construct

$$v^i_{(i)(j)} = H^i_{ijk} v^k = 0. \tag{4.21}$$

Therefore, commutating the formula (4.20) with respect to the indices j and i and using the equations (1.2) and (4.21), we have

$$(n-1) \sigma \psi_j = - v^s H^i_{sji} . \tag{4.22}$$

Differentiating covariantly the above formula with respect to x^m and taking notice of the equations (1.10), the latter part of (1.12) and (4.16) we get

$$-(\lambda_m + \psi_m) v^s H^i_{sji} - \sigma H^i_{mji} = (n-1) \sigma \psi_{j(m)} . \tag{4.23}$$

With the help of the equations (4.22) and (4.23), we can obtain

$$(\lambda_m + \psi_m)(n-1) \sigma \psi_j - \sigma H^i_{mji} = (n-1) \sigma \psi_{j(m)} . \tag{4.24}$$

Now, transvecting the above result by v^j and taking care of the equation (1.4) and (4.3), we find

$$\sigma H^i_{mij} v^j = (n-1) \sigma \psi_{j(m)} v^j . \tag{4.25}$$

In view of the equation (2.1), the last formula takes the form

$$\sigma v^i_{(m)(i)} = (n-1) \sigma \psi_{j(m)} v^j \tag{4.26}$$

or

$$v^i_{(m)(i)} = (n-1) \psi_{j(m)} v^j , \tag{4.27}$$

where we have neglected the non-vanishing $\sigma(x)$.

Introducing (4.20) into the left-hand side of the above result, we have

$$(n-1)\,\sigma\,\psi_m = (n-1)\,\psi_{j\,(m)}\,v^j \tag{4.28}$$

or

$$\sigma\,\psi_m = \psi_{j\,(m)}\,v^j = (\psi_j\,v^j)_{(m)} - \psi_j\,v^j{}_{(m)} = -\,\psi_j\,(\sigma\,\delta_m{}^j + \psi_m\,v^j) = -\,\sigma\,\psi_m \tag{4.29}$$

or

$$2\,\sigma\,\psi_m = 0. \tag{4.30}$$

Hence, by virtue of $\sigma \neq 0$, we get $\psi_m = 0$. This completes the proof of desired property. Thus, in this case the equation (1.12) becomes

$$\overline{x}^i = x^i + v^i(x)\,dt, \quad v^i{}_{(j)} = \sigma\,\delta_j{}^i \qquad (\sigma = \text{const.}). \tag{4.31}$$

This is a concurrent form and such a motion is unable to exist in a general PRFn-space. This fact was shown by the author in the paper [5].

2. PROJECTIVE AFFINE MOTION GIVEN BY (4.12)

From the equations (1.12) and (4.3) we can conclude

$$\psi_{h\,(m)}\,v^h + \psi_h\,\psi_m\,v^h + \sigma\,\psi_m = 0. \tag{4.32}$$

Subtracting the last formula with (2.7), we have

$$\psi_{h\,(m)}\,v^h = \psi_{m\,(h)}\,v^h. \tag{4.33}$$

By virtue of the equations (1.4) and (1.6), the formula (2.10) can be re-written as

$$-\,H_{jh}\,v^h = (n-1)\,\sigma\,\psi_j + \sigma_j. \tag{4.34}$$

Differentiating covariantly with respect to x^s and using the equations (1.10) and the latter part of (1.12), we get

$$-\,(\psi_s + \lambda_s)\,H_{jh}\,v^h - \sigma\,H_{js} = (n-1)\,\{\sigma_s\,\psi_j + \sigma\,\psi_{j\,(s)} + \sigma_{j\,(s)}\}. \tag{4.35}$$

Commutating the above formula with respect to the indices j and s and taking care of the fact that $(\sigma_{j\,(s)} = \sigma_{s\,(j)})$, we obtain

$$\begin{aligned} &-\,(\psi_s + \lambda_s)\,H_{jh}\,v^h - \sigma\,H_{js} - (n-1)\,(\sigma_s\,\psi_j + \sigma\,\psi_{j\,(s)}) = \\ &= -\,(\psi_j + \lambda_j)\,H_{sh}\,v^h - \sigma\,H_{sj} - (n-1)\,(\sigma_j\,\psi_s + \sigma\,\psi_{s\,(j)}). \end{aligned} \tag{4.36}$$

Now, transvecting the above formula by v^j and remembering the relations (2.8) and (4.3), we find

$$\begin{aligned} &-\,(\psi_s + \lambda_s)\,H_{jh}\,v^j\,v^h - \sigma\,H_{js}\,v^j - (n-1)\,\sigma\,\psi_{j\,(s)}\,v^j = \\ &= -\,\lambda_j\,v^j\,H_{sh}\,v^h - \sigma\,H_{sj}\,v^j - (n-1)\,\sigma\,\psi_{s\,(j)}\,v^j. \end{aligned} \tag{4.37}$$

By virtue of (4.33) and the hypothesis (4.12), the above equation reduces to

$$-(\psi_s+\lambda_s)\, H_{jh}\, v^j\, v^h - \sigma\, H_{js}\, v^j = \sigma H_{sj}\, v^j . \tag{4.38}$$

In view of the equations (1.2), (2.1), (2.3), (2.8) and (4.3), we can deduce :

$$\begin{aligned} - H_{jh}\, v^j\, v^h &= - H^s{}_{jhs}\, v^j\, v^h = - (v^s{}_{(h)\,(s)} - v^s{}_{(s)\,(h)})\, v^h = \\ &= - [(\sigma\, \delta_h^s + \psi_h\, v^s)_{(s)} - (n\sigma + \psi_s\, v^s)_{(h)}]\, v^h = \\ &= - [\sigma_h + \psi_{h(s)}\, v^s + \psi_h\, (n\sigma + \psi_s\, v^s) - n\sigma_h]\, v^h = \\ &= - \psi_{h(s)}\, v^h\, v^s = (\psi_h\, \psi_s\, v^s + \psi_h)\, v^h = 0. \end{aligned} \tag{4.39}$$

Thus, introducing the last result into the left-hand side of the formula (4.38) and neglecting the non-zero $\sigma(x)$, we have

$$(H_{js}+H_{sj})\, v^j = 0 \qquad (\sigma \neq 0). \tag{4.40}$$

Differentiating the above formula covariantly with respect to x^m and taking notice of the equations (1.10), (1.12) and (4.40) itself, we get

$$H_{ms}+H_{sm} . \tag{4.41}$$

On the other hand contracting the Bianchi's identity (1.5) for the Berwald's curvature tensor $H^i{}_{hjk}(x, \dot{x})$ with respect to the indices i and k, we obtain

$$H^i{}_{hji} + H^i{}_{jih} + H^i{}_{ihj} . \tag{4.42}$$

By virtue of the above identity and (1,6), we find

$$2\, H^i{}_{jih} + H^i{}_{ihj} = 0. \tag{4.43}$$

From the last formula, we can also have

$$(2H^i{}_{jih} + H^i{}_{ihj})\, v^j = 0. \tag{4.44}$$

In view of the equations (2.1 b) and (2.3), we can construct

$$H^i{}_{jih}\, v^j = - v^j\, H^i{}_{jhi} = v^i{}_{(h)\,(i)} - v^i{}_{(i)\,(h)} = (n-1)\,(\sigma\, \psi_h - \sigma_h) . \tag{4.45}$$

Next, introducing the above result into (4.44), we get

$$2\,(n-1)\,(\sigma\, \psi_h - \sigma_h) + v^i{}_{(i)\,(h)} = 0 \tag{4.46}$$

or

$$2\,(n-1)\,(\sigma\, \psi_h - \sigma_h) + (n\varepsilon + \phi_i\, v^i)_{(h)} = 0. \tag{4.47}$$

Thus, we obtain

$$\psi_h = \frac{(n-2)}{2\,(n-1)}\, \frac{\sigma_h}{\sigma} = \frac{(n-2)}{2\,(n-1)}\, (\partial_h \log \sigma), \tag{4.48}$$

namely, we have found at last that ψ_h denotes a gradient vector. In this way, in the present case, we could find that the motion (1.12) under consideration be a concircular projective affine motion. Such a motion has been investigated in detail by the author in [6].

5. THE CASE OF $\lambda_s v^s + 2\psi_s v^s + 2\sigma = 0$

In this case the equation (2.13) can be written as

$$\sigma_j (\psi_s v^s + \lambda_s v^s + 2\sigma) = 0 . \tag{5.1}$$

So, we have two cases:

$$\sigma_j = 0 \quad \text{or} \quad \sigma(x) = \text{non-zero const.}, \tag{5.2}$$

$$\psi_s v^s + \lambda_s v^s + 2\sigma = 0 . \tag{5.3}$$

But in view of the case (2.15 c), the condition (5.3) reduces to

$$\psi_s v^s = 0 \tag{5.4}$$

and such a case has been discussed in § 4. So, we except this case.

Thus in what follows, we shall study the possibility of projective affine motion under (5.2). By virtue of this case the formula (2.12) becomes

$$(n-1)\, \psi_j (\psi_s + \lambda_s) + H^i{}_{jis} = (n-1)\, \psi_{j(s)} , \tag{5.5}$$

where we have neglected the non-zero $\sigma(x)$.

With the help of the last equation we get the following commutation equality:

$$(n-1)\, \psi_j (\psi_s + \lambda_s) + H^i{}_{jis} - (n-1)\, \psi_s (\psi_j + \lambda_j) - H^i{}_{sij} = (n-1)\, (\psi_{j(s)} - \psi_{s(j)}). \tag{5.6}$$

From the identity (1.5) we can deduce

$$H^i{}_{jis} - H^i{}_{sij} - H^s{}_{isj} = 0. \tag{5.7}$$

Thus, by virtue of the above identity the formula (5.6) takes the form

$$H^i{}_{ijs} = (n-1)\, (\psi_{j(s)} - \psi_{s(j)} - \psi_j \lambda_s + \psi_s \lambda_j) . \tag{5.8}$$

Introducing the latter part of (1.12) into the equation (2.2), we have

$$(v^s \lambda_s + 2\sigma)\, H^i{}_{hjk} - \psi_s v^i H^s{}_{hjk} + \psi_h v^s H^i{}_{sjk} + \psi_j v^s H^i{}_{hsk} + \psi_k v^s H^i{}_{hjs} = 0. \tag{5.9}$$

Now, contracting the above formula with respect to the indices i and h we get

$$(v^s \lambda_s + 2\sigma)\, H^i{}_{ijk} + \psi_j v^s H^i{}_{isk} + \psi_k v^s H^i{}_{ijs} = 0. \tag{5.10}$$

On the other hand, introducing (5.8) into the left-hand side of the above equality and neglecting $(n-1) \neq 0$, we obtain

$$(v^s \lambda_s + 2\sigma)(\psi_{j(k)} - \psi_{k(j)} - \psi_j \lambda_k + \psi_k \lambda_j) + \psi_j v^s (\psi_{s(k)} - \psi_{k(s)} - \psi_s \lambda_k + \psi_{(k)} \lambda_s) + \\ + \psi_k v^s (\psi_{j(s)} - \psi_{s(j)} - \psi_j \lambda_s + \psi_s \lambda_j) = 0. \tag{5.11}$$

Now, introducing the hypothesis (2.15 c) into the above formula, we find

$$-2\psi_s v^s (\psi_{j(k)} - \psi_{k(j)} - \psi_j \lambda_k + \psi_k \lambda_j) + \psi_j v^s (\psi_{s(k)} - \psi_{k(s)} - \psi_s \lambda_k + \psi_k \lambda_s) + \\ + \psi_k v^s (\psi_{j(s)} - \psi_{s(j)} - \psi_j \lambda_s + \psi_s \lambda_j) = 0. \tag{5.12}$$

After little simplification, the above relation can also be re-written as

$$\psi_s v^s [-2(\psi_{j(k)} - \psi_{k(j)}) + \psi_j \lambda_k - \psi_k \lambda_j] + \psi_j v^s (\psi_{s(k)} - \psi_{k(s)}) + \\ + \psi_k v^s (\psi_{j(s)} - \psi_{s(j)}) = 0. \tag{5.13}$$

In view of the equation (2.7) the last formula yields

$$\psi_s v^s [-2(\psi_{j(k)} - \psi_{k(j)}) + \psi_j \lambda_k - \psi_k \lambda_j] + \psi_j v^s \psi_{s(k)} - \psi_k v^s \psi_{s(j)} - \\ - \psi_j (-\psi_s \psi_k v^s - \sigma \psi_k) + \psi_k (-\psi_j \psi_s v^s - \sigma \psi_j) = 0 \tag{5.14}$$

or

$$\psi_s v^s [-2(\psi_{j(k)} - \psi_{k(j)}) + \psi_j \lambda_k - \psi_k \lambda_j] + v^s (\psi_j \psi_{s(k)} - \psi_k \psi_{s(j)}) = 0. \tag{5.15}$$

Transvecting the above formula by v^k and taking care of (2.7), we have

$$\psi_s v^s (\psi_{k(j)} v^k - \psi_k v^k \lambda_j) - \psi_j \psi_k \psi_s v^s v^k - \sigma \psi_j \psi_s v^s = 0. \tag{5.16}$$

From the above result, neglecting the non-vanishing term $\psi_s v^s$, we get

$$\psi_{k(j)} v^k - \psi_k v^k (\lambda_j + \psi_j) + \sigma \psi_j = 0. \tag{5.17}$$

Now, subtracting the formula (2.7) with the last equation, we obtain

$$\psi_{k(j)} v^k - \psi_{j(k)} v^k - \lambda_j \psi_k v^k + \psi_j \lambda_k v^k = 0, \tag{5.18}$$

where we have used the hypothesis (2.15 c).

The above formula can also be re-written as

$$(\psi_{k(j)} - \psi_{j(k)} - \lambda_j \psi_k + \lambda_k \psi_j) v^k = 0. \tag{5.19}$$

Comparing the last relation with (5.8), we find

$$H^i_{ijs} v^j = 0 \tag{5.20}$$

or

$$v^i_{(i)(s)} = 0 \quad \text{or} \quad n\sigma_s + \psi_{i(s)} v^i + \sigma \psi_s + \psi_i \psi_s v^i = 0 \tag{5.21}$$

or

$$n\sigma_s + \psi_{i(s)} v^i + \psi_i (\sigma \delta_s^i + \psi_s v^i) = 0. \tag{5.22}$$

Since $\sigma = \text{const.}$ therefore the above formula reduces to

$$(\psi_i v^i)_{(\omega)} = 0 \quad \text{or} \quad \psi_i v^i = \text{const.}, \tag{5.23}$$

where we have used the latter part of (1.12). The last result can also be re-written as

$$\psi_{k(j)} v^k + \psi_k (\sigma \delta_j^k + \psi_j v^k) = 0. \tag{5.24}$$

Comparing the above equation with (2.7), we find

$$\psi_{j(k)} v^k = \psi_{k(j)} v^k . \tag{5.25}$$

Introducing the above relation in (5.18), we have

$$\lambda_j \psi_k v^k = \lambda_k \lambda_j v^k. \tag{5.26}$$

Since σ and $\psi_s v^s$ denotes a non-zero constant respectively, so from the hypothesis (2.15 c), we obtain

$$\lambda_s v^s = \text{const.} \tag{5.27}$$

From (5.26) we can put

$$\beta \lambda_j = \alpha \psi_j , \tag{5.28}$$

where

$$\text{a) } \beta \equiv \psi_k v^k = \text{non-zero const. and} \quad \text{b) } \alpha \equiv \lambda_k v^k = \text{const.} \tag{5.29}$$

Differentiating covariantly (5.20) by x^m and taking care of the equations (1.10), (1.12) and (5.20) itself, we get

$$\sigma H^i{}_{ims} = 0 . \tag{5.30}$$

For the non-vanishing property of the function $\sigma(x)$ the last formula yields

$$H^i{}_{ims} = 0. \tag{5.31}$$

By virtue of the last formula and the fact that $(n-1) \neq 0$, the relation (5.8) takes the form

$$\psi_{j(s)} - \psi_{s(j)} - \psi_j \lambda_s + \psi_s \lambda_j = 0 . \tag{5.32}$$

From (5.28), we can deduce

$$\beta (\lambda_{j(m)} - \lambda_{m(j)}) = \alpha (\psi_{j(m)} - \psi_{m(j)}). \tag{5.33}$$

In view of the equation (5.32), the above result reduces to

$$\beta (\lambda_{j(m)} - \lambda_{m(j)}) = \beta (\psi_j \lambda_m - \psi_m \lambda_j) . \tag{5.34}$$

With the help of (5.28) the above formula becomes

$$\beta(\lambda_{j(m)} - \lambda_{m(j)}) = \beta\lambda_j\lambda_m - \beta\lambda_m\lambda_j = 0 \tag{5.35}$$

or

$$\lambda_{j(m)} - \lambda_{m(j)} = 0\,. \tag{5.36}$$

Thus, we can say that λ_j is a gradient vector.

By virtue of (5.36) the formula (5.33) takes the form

$$\alpha(\psi_{j(m)} - \psi_{m(j)}) = 0\,. \tag{5.37}$$

Thus, from (5.28) and (5.37), we can consider the two cases :

i) **The case of $\alpha = 0$.** In this case we have $\lambda_j = 0$. Therefore from the fundamental definition (1.10), we can say that the space under consideration becomes symmetric (i.e. $H^i_{h,jk} = 0$).

ii) **The case of $\alpha \neq 0$.** In this case from (5.37), we can find

$$\psi_{j(m)} = \psi_{m(j)}\,. \tag{5.38}$$

Hence the transformation (1.7) characterized by (1.12) becomes one of the concircular form.

From all the discussions above we can state :

Theorem 5.1. If a general PRFn-space admits a projective affine motion of torse-forming form, the following three cases occur :

1. The case of $\sigma = 0$. In this case, the motion is degenerated into a recurrent motion ($n \geqslant 2$).

2. The case of $\psi_s v^s = 0$ and $\lambda_s v^s + 2\sigma = 0$. In this case only one projective affine motion of concircular form may be considered ($n \geqslant 3$).

3. The case of $\lambda_s v^s + 2\psi_s v^s + 2\sigma = 0$. In this case the motion must be a special one satisfying $\sigma =$ const. and furthermore we can regard the space reduces to a symmetric space or the motion can be degenerated into concircular motion ($n \geqslant 2$).

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ÖZET

Bu çalışmada, bir PRFn uzayında

$$\bar{x}^i = x^i + v^i(x)\, dt$$

ve $\pounds_u G^i_{jk} = 0$ ile karakterize edilen çeşitli tiplerdeki projektif afin hareketin varlığı incelenmektedir.