İstanbul Üniv. Fen Fak. Mec. Seri A, 43 (1978), 63 - 67

PROJECTIVE CURVATURE COLLINEATION AND SPECIAL PROJECTIVE MOTION IN A FINSLER SPACE

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In this paper it has been discussed the conditions under which a special projective motion becomes a projective curvature collineation.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space Fn [1][1]) having fundamental homogeneous metric function $F(x,\dot{x})$ of degree one in its directional arguments. The fundamental metric tensor $g_{ij}(x,\dot{x}) \overset{\text{def.}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x,\dot{x})$ of the space is symmetric in its lower indices. The covariant and contravariant metric tensors satisfy

$$g_{ik}\, g^{kj} = \delta_i^j\,, \tag{1.1}$$

where δ_i^j is Kronecker deltas. The projective covariant [4] derivative of a vector field $X^i(x,\dot{x})$ with respect to x^k is given by

$$X^i_{((k))} = \partial_k X^i - (\dot{\partial}_m X^i)\, \Pi_k^m + X^h\, \Pi^i_{hk}\,, \tag{1.2}$$

where

$$\Pi^i_{hk}(x,\dot{x}) \overset{\text{def.}}{=} \left\{ G^i_{hk} - \frac{1}{(n+1)} (2\, \delta^i_{(h}\, G^\gamma_{k)\gamma} + \dot{x}^i\, G^\gamma_{\gamma kh}) \right\} \tag{1.3}$$

are called projective connection coefficients and are also positively homogeneous function of degree zero in $\dot{x}^i$ and satisfy the following relations :

$$\text{a) } \Pi^i_{hk\gamma}\, \dot{x}^h = 0\,; \quad \text{b) } \dot{\partial}_j \Pi^i_{hk} = \Pi^i_{jhk} \text{ and c) } \Pi^i_{hk}\, \dot{x}^h = \Pi^i_k. \tag{1.4}$$

Also, like the Berwald's covariant derivative the projective covariant derivatives of $\dot{x}^i$ and δ_k^i vanishes.

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1) The numbers in square brackets refer to the references given at the end of the paper.

2) $\dot{\partial}_i \equiv \partial/\partial\dot{x}^i$ and $\partial_i \equiv \partial/\partial x^i$.

3) $2\, A_{(hk)} = A_{hk} + A_{kh}$ and $2\, A_{[hk]} = A_{hk} - A_{kh}$.

The following commutation formulae involving the projective covariant derivative are given by

$$\dot{\partial}_h (T_j^i{}_{((m))}) - (\dot{\partial}_h T_j^i)_{((m))} = T_j^\gamma \Pi^i_{\gamma hm} - T_\gamma^i \Pi^\gamma_{jhm} \tag{1.5}$$

and

$$2\, T^i_{j[((h))\,((k))]} = -(\dot{\partial}_\gamma T_j^i)\, Q^\gamma_{hk} + T_j^\gamma Q^i_{\gamma hk} - T_\gamma^i Q^\gamma_{jhk}\,; \tag{1.6}$$

where

$$Q^i_{jkh}(x,\dot{x}) \overset{\text{def.}}{=} 2\,\{\partial_{[h} \Pi^i_{k]j} - \Pi^i_{\gamma j[k} \Pi_{h]}^\gamma + \Pi_j^\gamma{}_{[k} \Pi^i_{h]\gamma}\} \tag{1.7}$$

are called projective entities and satisfy the following relations :

$$\text{a) } Q^i_{hk} = Q^i_{jhk}\,\dot{x}^j\,, \text{ b) } Q^i_k = Q^i_{kh}\,\dot{x}^k = Q^i_{jkh}\,\dot{x}^j\dot{x}^k \text{ and c) } Q^i_{jki} = Q_{jk}\,. \tag{1.8}$$

Let us consider an infinitesimal point transformation

$$\overline{x}^i = x^i + v^i(x)\,dt\,, \tag{1.9}$$

where $v^i(x)$ denotes components of contravariant vector and dt means an infinitesimal constant. When the point transformation (1.9) transforms the system of geodesies into the same system, (1.9) is called infinitesimal projective motion. The necessary and sufficient condition that (1.9) be a special projective motion in Fn is that the Lie-derivative of $\Pi^i_{jk}(x,\dot{x})$ with respect to (1.9) itself has the form

$$\pounds v\, \Pi^i_{jk}(x,\dot{x}) = \delta_j^i \sigma_k + \delta_k^i \sigma_j \tag{1.10}$$

for any non-zero covariant vector σ_k.

We have the following commutation formulae :

$$\pounds v\,(\dot{\partial}_k T_j^i) - \dot{\partial}_k (\pounds v\, T_j^i) = 0\,, \tag{1.11}$$

$$\pounds v\,(T^i_{j((k))}) - (\pounds v\, T_j^i)_{((k))} = T_j^h \pounds v\, \Pi^i_{kh} - T_h^i \pounds v\, \Pi^h_{kj} - (\dot{\partial}_h T_j^i) \pounds v\, \Pi^h_{ks}\,\dot{x}^s \tag{1.12}$$

and

$$\pounds v\,(\Pi^i_{jh})_{((k))} - (\pounds v\, \Pi^i_{kh})_{((j))} = \pounds v\, Q^i_{hjk} + (\pounds v\, \Pi^\gamma_{kl})\, \Pi^i_{\gamma jk}\,\dot{x}^l - (\pounds v\, \Pi^\gamma_{jl})\, \Pi^i_{k\gamma h}\,\dot{x}^l. \tag{1.13}$$

In view of (1.4b), (1.4c) and the commutation formula (1.11) the equation (1.10) yields two more equations

$$\pounds v\, \Pi^i_k(x,\dot{x}) = \dot{x}^i \sigma_k + \delta_k^i \sigma_j\,\dot{x}^j \tag{1.14}$$

and

$$\pounds v\, \Pi^i_{hjk}(x,\dot{x}) = \delta_j^i \sigma_{hk} + \delta_k^i \sigma_{hj}\,, \tag{1.15}$$

where

$$\text{a) } \sigma_{hk}(x,\dot{x}) \overset{\text{def.}}{=} \dot{\partial}_h \sigma_k \quad \text{and} \quad \text{b) } \sigma_h(x,\dot{x}) = \sigma_{hj}\, \dot{x}^j . \tag{1.16}$$

We shall now mention here some definitions which will be used in latter discussions.

Projective Curvature Collineation (Pande and Kumar [3]). A Finsler space is said to admit a projective curvature collineation if there exists a vector $v^i(x)$ such that

$$\pounds v\, Q^i_{hjk} = 0 . \tag{1.17}$$

Ricci Projective Curvature Collineation (Pande and Kumar [3]). An Fn is said to admit a Ricci projective curvature collineation if there exists a vector $v^i(x)$ such that

$$\pounds v\, Q_{hk} = 0 . \tag{1.18}$$

2. PROJECTIVE CURVATURE COLLINEATION

We shall now here discuss the conditions under which a special projective motion becomes a projective curvature collineation. In view of equation (1.10), the commutation formula (1.13) reduces to

$$\pounds v\, Q^i_{hjk}(x,\dot{x}) = 2\, \{\sigma_{h[((k))}\, \delta^i_{j]} + \delta^i_h\, \sigma_{[j((k))]}\} . \tag{2.1}$$

Introducing (1.17) in the above equation, we get

$$\sigma_{h[((k))}\, \delta^i_{j]} + \delta^i_j\, \sigma_{[j((k))]} = 0 . \tag{2.2}$$

Let us make a contraction with respect to indices i and j in (2.2), we obtain

$$\eta\, \sigma_{h((k))} - \sigma_{k(h)} = 0 . \tag{2.3}$$

Thus, we have :

Theorem 2.1. In a Finsler space Fn if a special projective motion becomes a projective curvature collineation, the equation (2.3) holds.

Contracting (2.1) with respect to the indices i and k and using (1.8c), we have

$$\pounds v\, Q_{hj} = \sigma_{j((h))} - n\, \sigma_{h((j))} , \tag{2.4}$$

where we have made use of the fact that the operations of contraction and Lie-differentiations are commutative. In view of (1.8), the equation (2.4) yields

$$\sigma_{j((h))} - n\, \sigma_{h((j))} = 0 , \tag{2.5}$$

which gives :

Theorem 2.2. In an Fn if special projective motion becomes a Ricci projective curvature collineation, the equation (2.5) holds.

3. PROJECTIVE SYMMETRIC SPACE

Definition 3.1. A Finsler space Fn is said to be a projective symmetric space if its projective covariant derivative of $Q^i_{hjk}(x,\dot{x})$ satisfies

$$Q^i_{hjk((\gamma))} = 0 \, , \tag{3.1}$$

which immediately yields

$$Q^i_{jk((\gamma))} = 0 \tag{3.2}$$

and

$$Q^i_{k((\gamma))} = 0 \, . \tag{3.3}$$

Using the commutation formula (1.12) to the projective entity $Q_j^i(x,\dot{x})$ and noting equations (1.10) and (3.3), we get

$$(\pounds v\, Q_j^i)_{((k))} = \{Q_k^i \sigma_j - Q_j^h \delta_k^i \sigma_h + \dot{x}^s (\dot{\partial}_k Q_j^i \sigma_s + \dot{\partial}_s Q_j^i \sigma_k)\} \, . \tag{3.4}$$

Contracting (3.4) with respect to indices i and j, we obtain

$$(\pounds v\, Q_i^i)_{((k))} = \dot{x}^s \{(\dot{\partial}_k Q_i^i)\sigma_s + (\dot{\partial}_s Q_i^i)\sigma_k\} \, . \tag{3.5}$$

Thus, we have:

Theorem 3.1. If a projective symmetric Finsler space admits a special projective motion, the equation (3.5) holds.

For the projective covariant derivative, the commutation formula is given by

$$\dot{\partial}_h (Q^i_{jk((m))}) - (\dot{\partial}_h Q^i_{jk})_{((m))} = Q^s_{jk}\,\Pi^i_{hms} - 2\, Q^i_{s[k}\,\Pi^s_{j]hm} \, , \tag{3.6}$$

which in view of equations (3.1) and (3.2) yields

$$Q^s_{jk}\,\Pi^i_{hms} - 2\, Q^i_{s[k}\,\Pi^s_{j]hm} = 0 \, . \tag{3.7}$$

In view of equation (1.8a) transvecting (2.1) by $\dot{x}^h$ and using the fact that $\pounds v\, \dot{x}^i = 0$, we get

$$\pounds v\, Q^i_{jk}(x,\dot{x}) = 2\, \{\sigma_{h[((k))}\, \delta^i_{j]}\, \dot{x}^h + \dot{x}^i\, \sigma_{h[j((k))]}\} \, . \tag{3.8}$$

In view of equations (1.4a), (1.15) and (3.8) taking the Lie-derivative of (3.7), we obtain

$$Q^s{}_{jk}\,\delta_m{}^i\,\sigma_{hs} + Q^i{}_{jk}\,\sigma_{hm} - 2\,\{Q^i{}_{h[k}\,\sigma_{j]|m} + Q^i{}_{m[k}\,\sigma_{j]h} - \\ - \{\sigma_{\gamma((s))}\,\dot{x}^\gamma\,\delta^i{}_{[k} - \dot{x}^i\,(\sigma_{s[((k))} - \sigma_{[k<((s))>})\}\,\Pi^i{}_{j]hm} = 0^{4)}\,. \tag{3.9}$$

Transvecting (3.9) by $\dot{x}^m$ and using equations (1.4a), (1.8b) and (1.16b), we get

$$Q^s{}_{jk}\,\sigma_{hs}\,\dot{x}^i + Q^i{}_{jk}\,\sigma_h - 2\,\{Q^i{}_{h[k}\,\sigma_{j]} + Q^i{}_{[k}\,\sigma_{j]h}\} = 0\,. \tag{3.10}$$

Thus, we have :

Theorem 3.3. If a projective symmetric Finsler space admits a special projective motion characterized by infinitesimal point transformation (1.9), the equation (3.10) holds.

Acknowledgement. The author expresses his sincere thanks to Dr. H.D.Pande for his constant encouragement and valuable guidance in preparing this paper.

4) The indices in $< >$ are free from symmetric and skew-symmetric parts.

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ÖZET

Bu çalışmada, özel bir projektif hareketin bir projektif eğrilik kolineasyonu olabilmesi şartları incelenmektedir.