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ON THE UNIFORM HARMONIC SUMMABILITY OF LEGENDRE SERIES

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In this paper it has been established a theorem for the uniform harmonic summability of the Legendre series.

The (C,α) and (N,q_n) summabilities of Legendre series have been discussed by a number of researchers like Hobson, Chapman, Haar, Plancherel, Kogbetliantz and Tripathi, but nothing seems to have been done so far in the direction of the study of Legendre series by uniform harmonic summability method. In attempt to make an advance in this direction, in the present paper we establish the following theorem for the uniform harmonic summability of the Legendre series.

1. The Legendre series, associated with a Lebesgue integrable function in the interval defined by $-1 \leqslant x \leqslant 1$ is

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x), \tag{1.1}$$

where

$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{+1} f(x) P_n(x)\, dx,$$

and the n-th Legendre polynomial $P_n(x)$ is defined by the following expansion:

$$\frac{1}{\sqrt{(1 - 2xZ + Z^2)}} = \sum_{n=0}^{\infty} P_n(x) Z^n .$$

We use the following notations :

$$\psi(t) = \psi(\theta,t) = f\{\cos(\theta - t)\} - f(\cos\theta) ;$$

$$\Psi(t) = \int_0^t |\psi(u)|\, du .$$

2. Let $u_0(x) + u_1(x) + u_2(x) + \dots$ be an infinite series, and

$$U_n(x) = u_0(x) + u_1(x) + u_2(x) + \dots + u_n(x). \tag{2.1}$$

Let $\{q_n\}$ be a sequence of constants, real or complex, and let

$$Q_n = q_0 + q_1 + \dots + q_n.$$

The sequence-to-sequence transformation

$$t_n(x) = \frac{1}{Q_n} \sum_{\nu=0}^{n} q_{n-\nu}\, u_\nu(x) \quad (Q_n \neq 0) \tag{2.2}$$

defines the sequence $\{t_n(x)\}$ of Nörlund means ([10], [15]) of the sequence $\{U_n(x)\}$ generated by the sequence of coefficients $\{q_n\}$.

If there exists a function $U = U(x)$, such that

$$\lim_{n\to\infty} [t_n(x) - U] = o(1) \tag{2.3}$$

uniformly in a set E, then we say that the series $\Sigma\, u_n(x)$ is summable (N, q_n) uniformly in E to the sum U.

The regularity conditions for (N, q_n) method are [4]

$$\lim_{n\to\infty} \frac{q_n}{Q_n} = 0 \tag{2.4}$$

and

$$\sum_{k=0}^{n} |q_k| = O(|Q_n|). \tag{2.5}$$

If it is assumed that $\{q_n\}$ is real, non-negative and monotonie non-increasing sequence, then the transformation defined by (2.2) is regular. Also the regularity of the method (N, q_n) implies that $Q_{n-1} / Q_n \to 1$ as $n \to \infty$.

Let us write

$$q(x) = \sum_{n=0}^{\infty} q_n x^n, \qquad Q(x) = \sum_{n=0}^{\infty} Q_n x^n.$$

Now for $|x| < 1$

$$\sum_{n=0}^{\infty} Q_n x^n \text{ and } \sum_{n=0}^{\infty} q_n x^n \equiv (1-x) \sum_{n=0}^{\infty} Q_n x^n$$

are convergent.

For $x \geqslant 1$, we extend the functions $q(x)$ and $Q(x)$ by linear interpolation to monotonic functions, which for $x = n$ $(n = 1,2,\ldots)$ take the values $q(n) = q_n$ and $Q(n) = Q_n$ respectively.

In the special case in which

$$q_n = \frac{1}{n+1}, \tag{2.6}$$

$$Q_n \sim \log n$$

as $n \to \infty$, the Nörlund mean reduces to harmonic mean, consequently (2.3) is equivalent to

$$\frac{1}{\log n} \sum_{k=0}^{n} \left(\frac{1}{k+1}\right) \left\{ U_{n-k}(x) - U \right\} = o(1) \tag{2.7}$$

uniformly in a set E, in which $U = U(x)$ is bounded and we shall say that the series $\sum_{\nu=0}^{\infty} u_\nu$ is summable by harmonic means uniformly in E to the sum U.

3. The Cesàro and Nörlund summabilities of Legendre series have been discussed by a number of researchers like Hobson [6], Chapman [1], Haar [1], Plancherel [11], Kogbetliantz [8] and Tripathi [14] but nothing seems to have been done so far in the direction of the study of Legendre series by uniform harmonic summability method. To make an advance in this direction, in the present paper we establish the following theorem for the uniform harmonic summability of Legendre series.

Theorem. If $\alpha(t)$ denotes a function of t, $\alpha(t)$ and $t / \alpha(t)$ ultimately increases steadly with t,

$$\log n = O\left[\alpha(Q_n)\right], \tag{3.1}$$

and

$$\int_0^t |f(x \pm u) - f(x)|\, dx = o\left[\frac{t}{\alpha(Q_\tau)}\right], \text{ as } t \to 0, \tag{3.2}$$

where $\tau = \left[\frac{1}{t}\right]$ is the integral part of $\frac{1}{t}$.

Uniformly in a set E defined in the interval $(-1, +1)$ in which $f(x)$ is bounded as $t \to +0$ then the series (1.1) is summable by harmonic means uniformly in E to the sum $f(x)$.

4. Following lemma are used in proving our theorem:

Lemma 1 [12].

$$\sum_{\nu=1}^{n} (2\nu + 1)\, P_\nu(x)\, P_\nu(y) = (n+1)\, \frac{P_{n+1}(y)\, P_n(x) - P_n(y)\, P_{n+1}(x)}{(y - x)} \,. \tag{4.1}$$

This identity is known as Christoffel's formula of summation.

Lemma 2. Under the condition (3.2) we have

$$\int_0^t |f\{\cos(\theta - \nu)\} - f(\cos\theta)\,|\, d\nu = o\left[\frac{t}{\alpha(Q_\tau)}\right] \quad \text{as } t \to 0, \tag{4.2}$$

where $x = \cos\theta$, $x + u = \cos\varphi$ and $\theta - \varphi = \nu$.

The proof of this lemma follows on the lines of *Foà* [2].

Lemma 3 [5]. If $0 < t \leqslant \pi$, then

$$\left| \sum_{k=0}^{n} \left(\frac{1}{k+1}\right) \cos(k+1)\, t \right| < A\left[1 + \log^+\left(\frac{1}{t}\right)\right], \tag{4.3}$$

where A is an absolute constant.

Lemma 4 [13]. For all values of n and t

$$\left| \sum_{k=0}^{n} \left(\frac{1}{k+1}\right) \sin(k+1)\, t \right| \leqslant \left(\frac{\pi}{2}\right) + 1\,. \tag{4.4}$$

Lemma 5 [19]. For $0 \leqslant a < b \leqslant \infty$, $0 < t \leqslant \pi$ and any n

$$\left| \sum_{k=a}^{b} q_n\, e^{i(n-k)t} \right| < c\, Q_\tau\,, \tag{4.5}$$

where c is an absolute constant.

5. Proof of theorem. The n-th partial sum of the series (1.1) is

$$S_n(x) = \sum_{\nu=0}^{n} a_\nu P_\nu(x)$$

$$= \frac{(n+1)}{2} \int_{-1}^{+1} f(y) \frac{P_{n+1}(y) P_n(x) - P_n(y) P_{n+1}(x)}{(y-x)} dy$$

by lemma 1.

Putting $f(y) \equiv 1$, it can be easily seen that

$$1 = \frac{(n+1)}{2} \int_{-1}^{+1} \frac{P_{n+1}(y) P_n(x) - P_n(y) P_{n+1}(x)}{(y-x)} dy.$$

Therefore

$$S_n(x) - f(x) = \frac{(n+1)}{2} \int_{-1}^{+1} [f(y) - f(x)] \frac{P_{n+1}(y) P_n(x) - P_n(y) P_{n+1}(x)}{(y-x)} dy,$$

and so

$$S_{n-k}(x) - f(x) =$$

$$= \frac{(n-k+1)}{2} \int_{-1}^{+1} [f(y) - f(x)] \frac{P_{n-k+1}(y) P_{n-k}(x) - P_{n-k}(y) P_{n-k+1}(x)}{(y-x)} dy.$$

Let us take a positive number S, less than 1 and consider it as the sum of the two other positive numbers μ and δ. Let d be another positive number, such that $0 < d < \mu$, and μ_x and $\mu_{x'}$ be two continuous functions of x within $(-1, +1)$, which lie within the limits $d \leqslant \mu_x \leqslant \mu$, $d \leqslant \mu_{x'} \leqslant \mu$.

Therefore, for $-1 + S \leqslant x \leqslant 1 - S$, we have

$$S_{n-k}(x) - f(x) = \frac{(n-k+1)}{2} \left[\int_{-1}^{x-\mu_x} + \int_{x-\mu_x}^{x+\mu_{x'}} + \int_{x+\mu_{x'}}^{+1} \right] \{f(y) - f(x)\} \times$$

$$\times \frac{P_{n-k+1}(y) P_{n-k}(x) - P_{n-k}(y) P_{n-k+1}(x)}{(y-x)} dy$$

$$= A_{n-k}(x) + B_{n-k}(x) + C_{n-k}(x), \tag{5.1}$$

say.

Hobson [7] has shown that uniformly for $-1+S \leqslant x \leqslant 1-S$,

$$\lim A_{n-k}(x) = 0\,, \tag{5.2}$$

and

$$\lim C_{n-k}(x) = 0\,, \tag{5.3}$$

uniformly in the set E.

Now we suppose $x = \cos\theta$, $y = \cos\varphi$, $0 < \theta < \pi$, $0 < \varphi < \pi$, $1-\delta = \cos\rho$, $1-(\mu+\delta) = 1-S = \cos(\rho+\sigma)$, $0 < \rho < \dfrac{\pi}{2}$, $0 < \sigma$, $\rho+\sigma < \dfrac{\pi}{2}$. Thus if η denotes the minimum of

$$[\arccos u - \arccos(u+\mu)]$$

for u in $(-1, 1-\mu)$, we have on the lines of Sansone [12]

$$B_{n-k}(\cos\theta) = \frac{(n-k+1)}{2}\int_{\theta-\eta}^{\theta+\eta}\{f(\cos\varphi) - f(\cos\theta)\} \times$$

$$\times \frac{[P_{n-k+1}(\cos\varphi)\,P_{n-k}(\cos\theta) - P_{n-k}(\cos\varphi)\,P_{n-k+1}(\cos\theta)]}{(\cos\varphi - \cos\theta)}\sin\varphi\, d\varphi\,,$$

in which $\rho+\sigma \leqslant \theta \leqslant \pi-(\rho+\sigma)$; $0 < \eta \leqslant \sigma$.

With successive transformation we obtain

$$B_{n-k}(\cos\theta) = D_{n-k}(\theta) + E_{n-k}(\theta)\,, \tag{5.4}$$

say, where

$$D_{n-k}(\theta) = \frac{1}{2\pi\sqrt{\sin\theta}}\int_{\theta-\eta}^{\theta+\eta}\frac{f(\cos\varphi)-f(\cos\theta)}{\sin\frac{1}{2}(\theta-\varphi)}\sin(n-k+1)(\theta-\varphi)\sqrt{\sin\varphi}\,d\varphi\,,$$

and obviously

$$E_{n-k}(\theta) = o(1)\,, \tag{5.5}$$

as $n \to \infty$, uniformly when x lies within $(-1+S, 1-S)$ *i.e.* in the set E.

Putting $(\theta-\varphi) = t$, we get

$$D_{n-k}(\theta) = \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\eta} \frac{f\{\cos(\theta - t)\} - f(\cos\theta)}{\sin\frac{1}{2}t} \times$$

$$\times \sin(n - k + 1)\, t \sqrt{\sin(\theta - t)}\, dt\,, \tag{5.6}$$

so we have from (5.1) to (5.6)

$$S_{n-k}(x) - f(x) = \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\eta} \frac{f\{\cos(\theta - t)\} - f(\cos\theta)}{\sin\frac{1}{2}t} \times$$

$$\times \sin(n - k + 1)\, t \sqrt{\sin(\theta - t)}\, dt + o\,(1)\,.$$

Since $f(x)$ is bounded on the set E, so $o(1)$ will tend to zero for any x uniformly in E.

Now

$$\frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(k+1)} \{S_{n-k}(x) - f(x)\} =$$

$$= \frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(k+1)} \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\eta} \frac{f\{\cos(\theta - t)\} - f(\cos\theta)}{\sin\frac{1}{2}t} \times$$

$$\times \sin\, (n - k + 1)\, t \sqrt{\sin(\theta - t)}\, dt + o\,(1)$$

uniformly in E,

$$= \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\eta} [f\{\cos(\theta - t)\} - f(\cos\theta)] \sqrt{\sin(\theta - t)} \times$$

$$\times \frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(k+1)} \frac{\sin(n - k + 1)\, t}{\sin\frac{1}{2}t}\, dt + o\,(1)\,,$$

uniformly in E,

$$= O\left[\int_0^{\eta} |\psi(t)|\, |N_n(t)|\, dt\right] + o\,(1)\,,$$

uniformly in E,

$$= O\left[\int_0^{n^{-1}} |\psi(t)|\, |N_n(t)|\, dt\right] + O\left[\int_{n^{-1}}^{\eta} |\psi(t)|\, |N_n(t)|\, dt\right] + o\,(1)\,,$$

uniformly in E,

$$= O\,(\mathrm{I}_1) + O\,(\mathrm{I}_2) + o\,(1)\,, \tag{5.7}$$

say, where

$$N_n\,(t) = \frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(k+1)} \frac{\sin\,(n-k+1)\,t}{\sin \frac{1}{2}\,t}\,.$$

In order to prove our theorem, we have to show that, under our assumptions

$$I_1 = o(1) \quad \text{and} \quad I_2 = o(1)$$

as $n \to \infty$, uniformly in the set E.

Now uniformly in $0 < t \leqslant \dfrac{1}{n}$

$$N_n\,(t) = O\,(n)\,,$$

so

$$\begin{aligned} I_1 &= \int_0^{n^{-1}} \psi\,(t)\, N_n\,(t)\, dt \\ &= O\left[n \int_0^{n^{-1}} |\,\psi\,(t)\,|\, dt \right] \\ &= o\left[n \frac{1}{n} \frac{1}{\alpha\,(Q_n)} \right] \\ &= o(1) \end{aligned} \tag{5.8}$$

as $n \to \infty$ uniformly in E.

In order to show that $I_2 = o(1)$, uniformly in E, we require a suitable estimation for the kernel $N_n\,(t)$, in the interval $(n^{-1}\,, \eta)$.

Since

$$N_n\,(t) = \frac{1}{\log n \sin \frac{1}{2}\,t} \left[\sin\,(n+2)\,t \sum_{k=0}^{n-1} \frac{\cos\,(k+1)\,t}{k+1} - \right.$$

$$\left. - \cos\,(n+2)\,t \sum_{k=0}^{n} \frac{\sin\,(k+1)\,t}{k+1} \right],$$

by using lemma 3 and 4, we get

$$\begin{aligned} N_n\,(t) &= O\left[\frac{1}{t \log n} \left\{ 1 + \log^+ \left(\frac{1}{t} \right) \right\} \right] \\ &= O\left[\frac{1}{t \log n} \log \left(\frac{1}{t} \right) \right], \end{aligned}$$

because $\log^+ \left(\dfrac{1}{t} \right) = \log \left(\dfrac{1}{t} \right)$ for every $t \in (n^{-1}\,, \eta)$.

Therefore

$$I_2 = \int_{n^{-1}}^{\eta} \psi(t)\, N_n(t)\, dt =$$

$$= O\left[\int_{n^{-1}}^{\eta} |\psi(t)|\, |N_n(t)|\, dt\right] =$$

$$= O\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta} |\psi(t)| \left|\frac{1}{t}\log\frac{1}{t}\right| dt\right] =$$

$$= O\left[\frac{1}{\log n}\left\{\psi(t)\,\frac{1}{t}\log\frac{1}{t}\right\}_{n^{-1}}^{\eta}\right] + O\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta} \psi(t)\,\frac{\left(1+\log\frac{1}{t}\right)}{t^2}\, dt\right]$$

$$= J_1 + J_2 \text{ say.}$$

Now

$$J_1 = O\left[\frac{1}{\log n}\left\{\psi(\eta)\,\frac{1}{\eta}\log\frac{1}{\eta}\right\}\right] + O\left[\frac{1}{\log n}\,\psi\left(\frac{1}{n}\right) n\log n\right] =$$

$$= J_{1\cdot 1} + J_{1\cdot 2} \text{ say,}$$

$$J_{1\cdot 1} = O\left[\frac{1}{\log n}\left\{\eta\cdot\frac{1}{\alpha\, Q\left(\frac{1}{\eta}\right)}\,\frac{1}{\eta}\,\alpha\, Q\left(\frac{1}{\eta}\right)\right\}\right]$$

$$= o\left[\frac{1}{\log n}\right]$$

$$= o(1)$$

as $n \to \infty$.

$$J_{1\cdot 2} = O\left[\frac{1}{\log n}\left\{\frac{1}{n}\,\frac{1}{\alpha(Q_n)}\, n\log n\right\}\right] =$$

$$= O\left[\frac{1}{\log n}\,\frac{\alpha(Q_n)}{\alpha(Q_n)}\right] =$$

$$= O\left[\left(\frac{1}{\log n}\right)\right] =$$

$$= o(1)$$

as $n \to \infty$. Therefore

$$J_1 = J_{1.1} + J_{1.2} =$$
$$= o(1) + o(1) =$$
$$= o(1).$$

Again

$$J_2 = O\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta}\frac{\psi(t)}{t^2}\,dt\right] + O\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta}\frac{\psi(t)\log\left(\frac{1}{t}\right)}{t^2}\,dt\right] =$$

$$= O\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta}\frac{t}{\alpha(Q_\tau)}\frac{1}{t^2}\,dt\right] + O\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta}\frac{t}{\alpha(Q_\tau)}\frac{1}{t^2}\log\left(\frac{1}{t}\right)dt\right] =$$

$$= o\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta}\frac{\log\left(\frac{1}{t}\right)}{t\log\left(\frac{1}{t}\right)\alpha(Q_\tau)}\,dt\right] + o\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta}\frac{1}{t}\frac{\alpha(Q_\tau)}{\alpha(Q_\tau)}\,dt\right] =$$

$$= O\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta^{-1}}\frac{\alpha(Q_\tau)}{\alpha(Q_\tau)}\frac{1}{t}\frac{1}{\log(1/t)}\,dt\right] + o\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta}\frac{1}{t}\,dt\right] =$$

$$= O\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta}\frac{1}{t\log\left(\frac{1}{t}\right)}\,dt\right] + o\left[\frac{1}{\log n}\int_{n^{-1}}^{\eta}\frac{1}{t}\,dt\right] =$$

$$= O\left[\left\{\frac{\log\log\frac{1}{t}}{\log n}\right\}_{n^{-1}}^{\eta}\right] + o\left[\left\{\frac{\log\left(\frac{1}{t}\right)}{\log n}\right\}_{n^{-1}}^{\eta}\right]$$

$$= o(1) \tag{5.9}$$

uniformly in E.

By virtue of (5.7) - (5.9) the theorem is completely proved.

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ÖZET

Bu çalışmada, Legendre serisinin düzgün harmonik toplanabilirliğine dair bir teorem ispat edilmektedir.