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PROJECTIVE BIANCHI AND VEBLEN IDENTITIES

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In this paper it has been found out the identities satisfied by the projective curvature tensor of Weyl.

1. Introduction. Let V_n be a Riemannian space of dimension n. Let us suppose that $R^h{}_{ijk}$ and R_{ij} be the Riemannian curvature and Ricci tensor respectively. The projective curvature tensor $W^h{}_{ijk}$ [1], defined by Weyl is given by

$$W^h{}_{ijk} \overset{\text{def.}}{=} R^h{}_{ijk} - \frac{1}{n-1}(\delta_k{}^h R_{ij} - \delta_j{}^h R_{ik}). \tag{1.i}$$

This tensor may be contracted in two ways, giving

$$W^h{}_{hjk} = 0 \tag{1.2}$$

and

$$W^h{}_{ijh} = 0. \tag{1.3}$$

2. Projective Bianchi identity and Bianchi tensor. Bianchi identities in a V_n are defined by

$$A^h{}_{ijkl} \overset{\text{def.}}{=} R^h{}_{ijk,l} + R^h{}_{ikl,j} + R^h{}_{ilj,k} = 0\,, \tag{2.1}$$

where (,) comma denotes the covariant differentiation with respect to x's.

Differentiating (1.1) covariantly with respect to x^1, we have

$$W^h{}_{ijk,l} = R^h{}_{ijk,l} - \frac{1}{(n-1)}(\delta_k{}^h R_{ij,l} - \delta_j{}^h R_{ik,l})\,. \tag{2.2}$$

On analogy of Bianchi identities for the curvature tensor of V_n we define the corresponding expression for the projective curvature tensor of V_n as follows:

$$U^h{}_{ijkl} = W^h{}_{ijk,l} + W^h{}_{ikl,j} + W^h{}_{ilj,k}\,; \tag{2.3}$$

With the help of (2.1) and (2.2) equation (2.3) becomes

$$U^h_{ijkl} = A^h_{ijk,l} - \frac{1}{(n-1)} [\delta_j^h (R_{li,k} - R_{ki,l}) + \\ + \delta_k^h (R_{ji,l} - R_{li,j}) + \delta_l^h (R_{ki,j} - R_{ji,k})] \,. \tag{2.4}$$

Contracting equation (2.4) for h and l, and on some simplification, we have

$$\frac{(n-1)}{(n-2)} (A^r_{ijlr} - U^r_{iljr}) = (R_{ji,l} - R_{li,j}) \,. \tag{2.5}$$

From (2.5), equation (2.4) becomes

$$U^h_{ijkl} = A^h_{ijkl} - \frac{1}{(n-2)} [\delta_j^h (A^r_{iklr} - U^r_{iklr}) + \\ + \delta_k^h (A^r_{iljr} - U^r_{iljr}) + \delta_l^h (A^r_{ijkr} - U^r_{ijkr})]. \tag{2.6}$$

Replacing δ_j^h by $g^{hm} g_{mj}$, etc. and making use of equations (1.2), (1.3) and (2.3), equation (2.6) becomes

$$W^h_{ijk,l} + W^h_{ikl,j} + W^h_{ilj,k} - \frac{g^{hm}}{(n-2)} [g_{mj} W^r_{ikl,r} + g_{mk} W^r_{ilj,r} + g_{ml} W^r_{ijk,r}] = \\ = A^h_{ijkl} - \frac{g^{hm}}{(n-2)} [g_{mj} A^r_{iklr} + g_{mk} A^r_{iljr} + g_{ml} A^r_{ijkr}] \,. \tag{2.7}$$

Since Bianchi identities in a V_n are satisfied, the right hand side of the equation (2.7) identically vanishes in view of equation (2.1). Let us put the left hand side of the equation (2.7) equal to K^h_{ijkl} so that

$$K^h_{ijkl} \stackrel{\text{def.}}{=} W^h_{ijk,l} + W^h_{ikl,j} + W^h_{ilj,k} - \\ - \frac{g^{hm}}{(n-2)} [g^{mj} W^r_{ikl,r} + g_{mk} W^r_{ilj,r} + g_{ml} W^r_{ijk,r}] = 0 \,. \tag{2.8}$$

We call (2.8) the 'Projective Bianchi identities' and the tensor K^h_{ijkl} as the 'Projective Bianchi tensor'.

From the last equation we may state the following theorem :

Theorem 2.1. In any Riemannian space V_n $(n > 2)$ the following projective Bianchi identities

$$K^h_{ijkl} = U^h_{ijkl} + P^h_{ijkl} \tag{2.9}$$

hold, where U^h_{ijkl} and P^h_{ijkl} are respectively the sum of first three terms and the remaining terms in the right hand side of (2.8).

We observe that the right hand side of (2.7) is also an identity. We call it the second form of Projective Bianchi identity and state the following :

Theorem 2.2. In any Riemannian space V_n $(n > 2)$ the following Bianchi identities in terms of the tensor $A^h{}_{ijkl}$ and $Q^h{}_{ijkl}$ hold and this is condensed to

$$K^h{}_{ijkl} = A^h{}_{ijkl} + Q^h{}_{ijkl}\,, \tag{2.10}$$

where $Q^h{}_{ijkl}$ is the right hand side of (2.7) excluding the first term.

Remark 2.1. If the covariant derivative of Ricci tensor is symmetric, then we have

$$R_{ij,k} = R_{kj,i}\,.$$

Now from equations (2.1), (2.3) and (2.4) and considering remark (2.1), we establish the following :

Theorem 2.3. The projective Bianchi identity and Bianchi identities in a V_n are identical is that the Ricci tensor is covariant constant.

Theorem 2.4. The projective Bianchi identity and Bianchi identities in a V_n are identical is that the covariant derivative of Ricci tensor is symmetric.

3. Projective Veblen Identity and Veblen Tensor. The Veblen identities in a V_n are defined as follows :

$$V^h{}_{ijkl} \overset{\text{def.}}{=} R^h{}_{ijk,l} + R^h{}_{kil,j} + R^h{}_{lkj,i} + R^h{}_{jli,k} = 0\,. \tag{3.1}$$

Now we shall find out the corresponding expression for the projective Veblen identities on analogy of (3.1) as follows :

$$T^h{}_{ijkl} \overset{\text{def.}}{=} W^h{}_{ijk,l} + W^h{}_{kil,j} + W^h{}_{lkj,i} + W^h{}_{jli,k}\,. \tag{3.2}$$

Substituting from (2.2) in (3.2) and using (3.1), we have

$$T^h{}_{ijkl} = V^h{}_{ijkl} - \frac{1}{(n-1)}\,[\delta_k{}^h (R_{ij,l} - R_{lj,i}) + \delta_i{}^h (R_{jl,k} - R_{kl,j}) +$$

$$+ \delta_j{}^h (R_{lk,i} - R_{ik,l}) + \delta_l{}^h (R_{ki,j} - R_{ji,k})]\,. \tag{3.3}$$

Contracting (3.3) for h and l, and on some simplification, we have

$$\frac{(n-1)}{(n-2)}\,[V^r{}_{ijkr} - T^r{}_{ijkr}] = (R_{kl,j} - R_{ji,k}). \tag{3.4}$$

From equation (3.4), equation (3.3) becomes

$$T^h{}_{ijkl} = V^h{}_{ijkl} - \frac{1}{(n-2)} [\delta_i^h (V^r{}_{lkjr} - T^r{}_{lkjr}) + \delta_j^h (V^r{}_{kilr} - T^r{}_{kilr}) +$$
$$+ \delta_k^h (V^r{}_{jlik} - T^r{}_{jlik}) + \delta_l^h (V^r{}_{ijkr} - T^r{}_{ijkr})]. \tag{3.5}$$

Replacing δ_i^h etc., by $g^{hm} g_{mi}$ etc. and making use of equations (1.2) and (1.3) in equation (3.5), we have

$$W^h{}_{ijk,l} + W^h{}_{kil,j} + W^h{}_{lkj,i} + W^h{}_{jli,k} - \frac{g^{hm}}{(n-2)} [g_{mi} W^r{}_{lkj,r} + g_{mj} W^r{}_{kil,r} +$$
$$+ g_{mk} W^r{}_{jli,r} + g_{ml} W^r{}_{ijk,r}] = V^h{}_{ijkl} - \frac{g^{hm}}{(n-2)} [g_{mi} V^r{}_{lkjr} + g_{mj} V^r{}_{kilr} +$$
$$+ g_{mk} V^r{}_{jlir} + g_{ml} V^r{}_{ijkr}]. \tag{3.6}$$

Since Veblen identities in a V_n are satisfied the right hand side of above equation is identically zero in view of (3.1). On putting the left hand side of equation (3.6) equal to $S^h{}_{ijk,l}$, we have

$$S^h{}_{ijkl} \overset{\text{def.}}{=} W^h{}_{ijk,l} + W^h{}_{kil,j} + W^h{}_{lkj,i} + W^h{}_{jli,k} - \frac{g^{hm}}{(n-2)} [g_{mi} W^r{}_{lkj,r} +$$
$$+ g_{mj} W^r{}_{kil,r} + g_{mk} W^r{}_{jli,r} + g_{ml} W^r{}_{ijk,r}] = 0. \tag{3.7}$$

We call (3.7) the 'Projective Veblen identity' and the tensor $S^h{}_{ijkl}$ as the 'Projective Veblen tensor'.

From equation (3.7) we may state the following :

Theorem 3.1. In any Riemannian space $V_n (n > 2)$ the following projective Veblen identities

$$S^h{}_{ijkl} = T^h{}_{ijkl} + B^h{}_{ijkl} \tag{3.8}$$

hold, where $T^h{}_{ijkl}$ and $B^h{}_{ijkl}$ are respectively the sum of four terms and remaining terms in the right hand side of equation (3.7).

We observe that the right hand side of (3.6) is also an identity. We call it the second form of Projective Veblen identity and state the following :

Theorem 3.2. In any Riemannian space $V_n (n > 2)$, the following Projective Veblen identity holds:

$$S^h{}_{ijkl} = V^h{}_{ijkl} + C^h{}_{ijkl}, \tag{3.9}$$

where $C^h{}_{ijkl}$ is the right hand side of (3.6) excluding the first term.

From equations (3.1), (3.2), (3.3) and remark 2.1, we establish the following:

Theorem 3.3. The Veblen and Projective Veblen identities in a V_n $(n > 2)$ are identical is that either of the following holds:

(a) covariant derivative of Ricci tensor is symmetric.

(b) Ricci tensor is covariant constant.

Proof: (*i*) From the remark 2.1 and from equation (3.3) the result (a) follows.

(*ii*) If the Ricci tensor is covariant constant then we have

$$R_{ij,k} = 0 .$$

Therefore from this and from equation (3.3) the result (b) follows.

From equations (3.7) and (2.8), we have

$$S^h{}_{ijkl} + S^h{}_{iklj} + S^h{}_{iljk} = K^h{}_{ijkl} + K^h{}_{kilj} + K^h{}_{lijk} + K^h{}_{jlik} + D^h{}_{ijkl}\,, \quad (3.10)$$

where

$$D^h{}_{ijkl} = 2\,(W^r{}_{klj,i} + W^h{}_{ljk,i} + W^h{}_{jkl,i}) -$$

$$- \frac{2\,g^{hm}}{(n-2)}\,[g_{mi}\,(W^r{}_{klj,r} + W^r{}_{ljk,r} + W^r{}_{jkl,r})].$$

Theorem 3.4. (3.10) is the relation between Projective Bianchi identity and Projective Veblen identities defined in (2.8) and (3.7) respectively.

On account of (3.10) we may establish:

Theorem 3.5. The necessary and sufficient condition that Projective Bianchi and Veblen identities may be expressed explicitly in terms of one another is that the tensor $D^h{}_{ijkl}$ is equal to zero.

4. Application to an Einstein space. In this section we shall study the Projective Bianchi identity and Projective Veblen identities in an Einstein space V_n. It is well known that every V_2 is an Einstein space. An Einstein space V_3 is a spherical space of constant Riemannian curvature [3]. Therefore we shall consider the case $n \geqslant 4$.

Let us suppose that V_n is an Einstein space, therefore, we have

$$R_{ij} = \frac{R}{n} g_{ij} . \tag{4.1}$$

For $n > 2$, the equation (4.1) will hold only if R is a constant [2]. Therefore by virtue of (4.1) equations (2.4) and (3.3) reduce to

$$U^h{}_{ijkl} = A^h{}_{ijkl} \tag{4.2}$$

and

$$T^h{}_{ijkl} = V^h{}_{ijkl} . \tag{4.3}$$

From (2.7) and (4.2) we have

$$\frac{g^{hm}}{(n-2)} [g_{mj} W^r{}_{ikl,r} + g_{mk} W^r{}_{ilj,r} + g_{ml} W^r{}_{ijk,r}] = 0 . \tag{4.4}$$

Since in a Riemannian space V_n, Bianchi identities are satisfied, therefore from (4.2) and (2.1) we have

$$U^h{}_{ijkl} = 0 . \tag{4.5}$$

From (4.5) and (4.2), we state the following:

Theorem 4.1. Projective Bianchi identities in an Einstein space and in a Riemannian space are identical.

Theorem 4.2. For an Einstein space, equations (4.4) and (4.5) are identities and these are equivalent to Projective Bianchi identities in this space.

From equation (4.2), we also have:

Theorem 4.3. Projective Bianchi identities in an Einstein space are equivalent to Bianchi identities in a Riemannian space V_n and one can be expressed explicitly in terms of the other.

Also from equations (3.6) and (4.3), we have

$$\frac{g^{hm}}{(n-2)} [g_{ml} W^r{}_{lkj,r} + g_{mj} W^r{}_{kil,j} + g_{mk} W^r{}_{jli,r} + g_{ml} W^r{}_{ijk,r}] = 0. \tag{4.6}$$

In a Riemannian space V_n, Veblen identities are satisfied. Therefore from (3.1) and (4.3), we have

$$T^h{}_{ijkl} = 0 . \tag{4.7}$$

Now from (4.7) and (4.3), we have:

Theorem 4.4. Projective Veblen identities in an Einstein space and in a Riemannian space are identical.

Theorem 4.5. For an Einstein space (4.6) and (4.7) are identities and these are equivalent to Projective Veblen identities in this space.

From (4.3), we may establish the following:

Theorem 4.6. Projective Veblen identities in an Einstein space are equivalent to ordinary Veblen identities in a Riemannian space V_n and one can be expressed explicitly in terms of the other.

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ÖZET

Bu çalışmada, Weyl projektif eğrilik tensörünün sağladığı bazı özdeşlikler elde edilmektedir.