İstanbul Üniv. Fen Fak. Mec. Seri A, 43 (1978), 131-138

PROJECTIVE AFFINE MOTION IN PR Fn-SPACE

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In this paper it has been investigated the case $\lambda_s v^s = 0$ occured when $\psi_h \neq$ gradient vector and hence completed authors earlier results on projective affine motions in PR Fn-space.

1. INTRODUCTION

Let us consider an n-dimensional affinelly connected and non flat Finsler space $[Fn]$[1] equipped with symmetric Berwald's connection coefficient $G^i_{jk}(x,\dot{x})$. The covariant derivative of any tensor field $T_j^i(x,\dot{x})$ with respect to x^k is given by

$$T^i_{j(k)} = \partial_k T_j^i - \dot{\partial}_h T_j^i G^h_{k\nu} \dot{x}^\nu + T_j^s G^i_{sk} - T_s^i G^s_{jk}\,{}^{2)} . \tag{1.1}$$

The well known commutation formula involving the above covariant derivative is given by

$$2\, T^i_{j[(h)(k)]} = -\dot{\partial}_\nu T_j^i H^\nu_{shk} \dot{x}^s + T_j^s H^i_{shk} - T_s^i H^s_{jhk}\,{}^{3)} , \tag{1.2}$$

where

$$H^i_{hjk}(x,\dot{x}) \overset{\text{def.}}{=\!=} 2\,\{\partial_{[k} G^i_{j]h} - G^i_{\nu h[j} G^\nu_{k]} + G^\nu_{h[j} G^i_{k]\nu}\} \tag{1.3}$$

is called Berwald's curvature tensor field and satisfies the following relations [1] :

$$H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0 , \tag{1.4}$$

$$H^i_{hji} = H_{hj} \tag{1.5}$$

and

$$H^i_{hjk} = -H^i_{hkj} . \tag{1.6}$$

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1) Numbers in brackets refer to the reference at the end of the paper.

2) $\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$ and $\partial_i \equiv \partial/\partial x^i$.

3) $2\, A_{[hk]} = A_{hk} - A_{kh}$.

In an Fn, if the Berwald's curvature tensor field $H^i_{hjk}(x,\dot{x})$ satisfies the following relation :

$$H^i_{hjk(s)} = \lambda_s H^i_{hjk}, \tag{1.7}$$

where λ_s is any vector field, then the space is called projective recurrent Finsler space or PR Fn-space and λ_s is called projective recurrence vector. Let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\, dt, \tag{1.8}$$

where $v^i(x)$ is any vector field and dt is an infinitesimal point constant. In view of (1.8) and Berwald's covariant derivative, the Lie derivatives of T_j^i and G^i_{jk} are given by [2]

$$\pounds v\, T_j^i(x,\dot{x}) = T^i_{j(h)} v^h + \dot{\partial}_h T_j^i v^h_{(s)} \dot{x}^s + T_s^i v^s_{(j)} - T_j^s v^i_{(s)} \tag{1.9}$$

and

$$\pounds v\, G^i_{jk} = v^i_{(j)(k)} - H^i_{jkh} v^h + G^i_{sjk} v^s_{(\nu)} \dot{x}^\nu . \tag{1.10}$$

In a PR Fn-space, when we consider an infinitesimal projective recurrent affine motion of the general form

$$\bar{x}^i = x^i + v^i(x)\, dt, \; v^i_{(h)} = \psi_h(x) v^i, \tag{1.11}$$

there exists a case characterized by

$$\psi_s v^s = \text{const.} \tag{1.12}$$

under the assumptions :

$$\text{a) } H_{hs} v^s = 0 \quad \text{and} \quad \text{b) } (\lambda_s + \psi_s) v^s \neq 0 . \tag{1.13}$$

By virtue of the equation (1.12) we are able to get the following relation :

$$(\psi_{h(k)} - \psi_{k(h)}) \lambda_s v^s = 0 . \tag{1.14}$$

From the above formula we can have two cases :

$$\text{a) } \lambda_s v^s = 0 \quad \text{and} \quad \text{b) } \psi_h = \text{gradient vector} . \tag{1.15}$$

The present author has showed these facts in his paper [7] and, under $\lambda_s v^s \neq 0$, discussed about (1.15b) deeply, but he has kept mum about the former case (1.15a) occurred when $\psi_h \neq$ gradient vector. So, in this short paper we shall try to investigate this case and bring our long study on projective affine motions in PR Fn-space to a close.

2. CHARACTERISTIC CONDITION

If a PR Fn-space admits an infinitesimal projective affine motion $\bar{x}^i = x^i + v^i(x)\,dt$ with respect to the vector field $v^i(x)$, we have

$$\pounds v\, G^i_{jk} = 0, \tag{2.1}$$

i.e.

$$v^i_{(j)(k)} = H^i_{jkh}\, v^h + G^i_{sjk}\, v^s_{(\nu)}\, \dot{x}^\nu \tag{2.2}$$

and its integrability condition

$$\pounds v\, H^i_{hjk} = 0\,. \tag{2.3}$$

In the present case, we assume (2.1) and (2.3) under which we devise to find the characteristic condition for the projective recurrent motion (1.11) with (1.15a).

In view of basic definition (1.9) the formula (2.3) reduces to

$$\begin{aligned}\pounds v\, H^i_{hjk} = H^i_{hjk(s)}\, v^s - H^s_{hjk}\, v^i_{(s)} + H^i_{sjk}\, v^s_{(h)} + H^i_{hsk}\, v^s_{(j)} + H^i_{hjs}\, v^s_{(k)} + \\ + \dot{\partial}_s H^i_{hjk}\, v^s_{(\nu)}\, \dot{x}^\nu = 0\,.\end{aligned} \tag{2.4}$$

In view of the latter part of the basic conditions (1.11), (1.7) and (2.2) equation can be written as

$$\begin{aligned}\pounds v\, H^i_{hjk} &= \lambda_s v^s H^i_{hjk} - H^s_{hjk}\, v^i\, \psi_s + H^i_{sjk}\, v^s\, \psi_h + H^i_{hsk}\, v^s\, \psi_j + H^i_{hjs}\, v^s\, \psi_k = 0\\ &= \lambda_s v^s - v^i\,(\psi_{h(j)(k)} - \psi_{h(k)(j)}) + \psi_h\,(v^i_{(j)(k)} - v^i_{(k)(j)}) - \psi_j\, v^i_{(h)(k)} + \\ &\qquad + \psi_k\, v^i_{(h)(j)} = 0\\ &= \lambda_s v^s H^i_{hjk} - v^i\,(\psi_{h(j)(k)} - \psi_{h(k)(j)}) + \psi_h\,(\psi_{j(k)} - \psi_{k(j)})\, v^i - \\ &\qquad - \psi_j\,(\psi_{h(k)} - \psi_h\,\psi_k)\, v^i + \psi_k\,(\psi_{h(j)} - \psi_h\,\psi_j)\, v^i = 0\\ &= \lambda_s v^s H^i_{hjk} - (\psi_{h(j)} + \psi_h\,\psi_j)_{(k)}\, v^i + (\psi_{h(k)} + \psi_h\,\psi_k)_{(j)}\, v^i = 0\\ &= \lambda_s v^s H^i_{hjk} - \{(\psi_{h(j)} + \psi_h\,\psi_j)_{(k)} - (\psi_{h(k)} + \psi_h\,\psi_k)\}\, v^i = 0\,,\end{aligned} \tag{2.5}$$

where we have used the commutation formula (1.2) in the process of calculation.

Consequently, if (1.15a) will be the case, we get here

$$(\psi_{h(j)} + \psi_h\,\psi_j)_{(k)} = (\psi_{h(k)} + \psi_h\,\psi_k)_{(j)}\,. \tag{2.6}$$

And, if we have (2.6) conversely from the last relation of (2.5), we get

$$\lambda_s\, v^s\, H^i_{hjk} = 0\,. \tag{2.7}$$

For a non-flat PR Fn-space the above relation yields (1.15a). Thus, we have :

Conclusion 2.1a. In a general PR Fn-space, when the equation of projective affine motion $\pounds v\, G^i_{jk} = 0$ is integrable, in order that the motion (1.11) have a form satisfying (1.15a), it is necessary and sufficient that we have the commutative condition (2.6).

Conclusion 2.1b. In a general PR Fn-space, in order that the equation $\pounds v\, G^i_{jk} = 0$ of projective affine motion (1.11) with (1.15a) is integrable, it is necessary and sufficient that we have the condition (2.6).

3. CONTINUED DISCUSSION

In an affinelly connected Finsler space the Bianchi's identity for Berwald's curvature tensor H^i_{hjk} reduces to

$$H^i_{hjk(s)} + H^i_{hks(j)} + H^i_{hsj(k)} = 0 . \tag{3.1}$$

By virtue of the basic definition (1.7), the last formula reduces to

$$\lambda_s H^i_{hjk} + \lambda_j H^i_{hks} + \lambda_k H^i_{hsj} = 0 . \tag{3.2}$$

Transvecting the above formula by v^s and using the hypothesis (1.15a), we find

$$\lambda_j H^i_{hks} v^s = \lambda_k H^i_{hjs} v^s , \tag{3.3}$$

where we have used (1.6).

By virtue of the basic condition (2.2) the above formula yields

$$\lambda_j (v^i_{(h)(k)} - G^i_{shk} v^s_{(v)} \dot{x}^v) = \lambda_k (v^i_{(h)(j)} - G^i_{shj} v^s_{(v)} \dot{x}^v). \tag{3.4}$$

Introducing the latter part of the fundamental assumption (1.11), we obtain

$$\lambda_j (\psi_{h(k)} + \psi_h \psi_k) v^i = \lambda_k (\psi_{h(j)} + \psi_h \psi_j) v^i . \tag{3.5}$$

For the non-vanishing $v^i(x)$ the above formula takes the form

$$\lambda_j (\psi_{h(k)} + \psi_h \psi_k) = \lambda_k (\psi_{h(j)} + \psi_h \psi_j) . \tag{3.6}$$

Conversely, if we have (3.3), multiply this by v^i and taking notice of (1.11), we have (3.4), i.e. (3.3). In view of (3.3) and the identity (3.2), we get $\lambda_s v^s H^i_{hjk} = 0$, from which we have $\lambda_s v^s = 0$. Thus, we have:

Conclusion 3.1. When a PR Fn-space admits a projective affine motion of the general recurrent form (1.11), in order to have (1.15a) always, it is necessary and sufficient that we have (3.6).

Thus upon the same fact, we have obtained two conclusions 2.1 and 3.1. In what follows, we shall discuss about the mutual relation existing between these conclusions. For this purpose, we shall begin our study from the second conclusion, i.e. (3.6).

The PR Fn-space under consideration is not a symmetric one, say so according to the assumption (3.6), we can suppose a suitable vector E_h such that

$$\psi_{h(k)} + \psi_h \psi_k = E_h \lambda_k . \tag{3.7}$$

In view of the latter part of (1.11), the above formula can also be re-written as

$$v^i_{(h)(k)} = E_h \lambda_k v^i . \tag{3.8}$$

Differentiating (3.2) covariantly with respect to x^m in the sense of Berwald and taking notice of the basic formulae (1.7) and (3.2) itself, we obtain

$$\lambda_{s(m)} H^i{}_{hjk} + \lambda_{j(m)} H^i{}_{hks} + \lambda_{k(m)} H^i{}_{hsj} = 0 . \tag{3.9}$$

Again differentiating the assumption (1.15a) covariantly with respect to x^m and using the latter part of (1.11) and (1.15a) itself, we get

$$\lambda_{s(m)} v^s + \lambda_s \psi_m v^s = 0 \tag{3.10}$$

or

$$\lambda_{s(m)} v^s = 0 . \tag{3.11}$$

Next, transvecting the identity (3.9) by v^k and taking care of the equations (2.2) and (3.11) we find

$$\lambda_{s(m)} (v^i{}_{(h)(j)} - G^i{}_{shj} v^s{}_{(\nu)} \dot{x}^\nu) = \lambda_{j(m)} (v^i{}_{(h)(s)} - G^i{}_{shk} v^s{}_{(\nu)} \dot{x}^\nu) . \tag{3.12}$$

Introducing (3.8) into the above relation, we have

$$(\lambda_{s(m)} \lambda_j - \lambda_{j(m)} \lambda_s) E_h = 0 , \tag{3.13}$$

where we have neglected non-zero v^i .

Thus, we get here two cases :

$$\text{a) } E_h = 0 \quad \text{and} \quad \text{b) } \lambda_{s(m)} \lambda_j = \lambda_{j(m)} \lambda_s . \tag{3.14}$$

4. ESSENTIAL DISCUSSION

The above all discussion stands only on the assumption $\lambda_s v^s = 0$ derived from $\psi_h v^h =$ const. under $\psi_{k(s)} \neq \psi_{s(k)}$. In the present case of $\lambda_s v^s = 0$, we shall show conversely that $\psi_h v^h =$ const. holds good.

Now, if (3.14a) will be the case, from (3.7) we can get

$$(\psi_{h(k)} + \psi_h \psi_k) v^h = (E_h v^h) \lambda_k = 0, \text{ i.e. } \psi_h v^h = \text{const.} \tag{4.1}$$

and $\psi_{h(k)} + \psi_h \psi_k = 0$. From the latter, we can find

$$\psi_{h(k)} = \psi_{k(h)} . \tag{4.2}$$

However, under $\psi_h v^h = \text{const.}$, we are considering only a case in which we have $\psi_{h(k)} \neq \psi_{k(h)}$ and $\lambda_s v^s = 0$, so (3.14a) must be an exceptional condition. Then let us consider (3.14b). In this case, at first, we have to test also the following fact : $\psi_h v^h = \text{const.}$ Before the discussion of this fact, we shall find a few important conditions. In such a case, we can suppose the existence of a covariant vector M_k such as

$$\lambda_{k(k)} = \lambda_h M_k . \tag{4.3}$$

Comparing the equations (2.2) and (3.8) and using the latter part of (1.11) we can deduce an equality :

$$H^i{}_{jkh} v^h = E_j \lambda_k v^i . \tag{4.4}$$

Differentiating the above formula covariantly with respect to x^m and taking care of the basic conditions (1.7) and the latter part of (1.11), we get

$$(\lambda_m + \psi_m) H^i{}_{jkh} v^h = E_{j(m)} \lambda_k v^i + E_j \lambda_{k(m)} v^i + E_j \lambda_k \psi_m v^i . \tag{4.5}$$

By virtue of the formula (4.3) and (4.4), the last relation takes the form :

$$(\lambda_m + \psi_m) E_j \lambda_k v^i = E_{j(m)} \lambda_k v^i + E_j \lambda_k M_m v^i + E_j \lambda_k \psi_m v^i . \tag{4.6}$$

Neglecting the non-zero terms λ_k and v^i from the last formula, we obtain

$$E_{j(m)} = (\lambda_m - M_m) E_j . \tag{4.7}$$

In view of the equations (3.7), (4.3) and (4.7), we can conclude the following interesting formula :

$$(\psi_{h(k)} + \psi_h \psi_k)_{(m)} = (E_h \lambda_k)_{(m)} = (\lambda_m - M_m) E_h \lambda_k + E_h \lambda_k M_m \tag{4.8}$$

or

$$(\psi_{h(k)} + \psi_h \psi_k)_{(m)} = E_h \lambda_k \lambda_m . \tag{4.9}$$

This yields the condition (2.6). By virtue of (3.7), the last formula can be written as

$$(\psi_{h(k)} + \psi_h \psi_k)_{(m)} = \lambda_m (\psi_{k(k)} + \psi_h \psi_k) . \tag{4.10}$$

Now, we shall discuss about the condition $\psi_h v^h = \text{const.}$ As the author has showed [7] when we consider a projective affine motion of projective reccurrent form (1.11), we have always [7] :

$$\text{a) } \pounds v\, \psi_h = 0, \quad \text{b) } \pounds v\, \lambda_h = 0 \quad \text{and} \quad \text{c) } \pounds v\, \psi_{h(k)} = (\pounds v\, \psi_h)_{(k)} . \tag{4.11}$$

On the other hand, operating $\pounds v$ to the both side of the equation (3.7), and noting the above identities, we have

$$\pounds v\, E_h = E_{h(k)} v^k + E_s v^s{}_{(h)} = E_{h(k)} v^k + E_s v^s \psi_h = 0 . \tag{4.12}$$

In view of the basic conditions (1.9), (1.11) and (4.11b), we can get

$$\lambda_{h(k)} v^k + \lambda_s v^s \psi_h = 0 . \tag{4.13}$$

From the basic assumption (1.15a), the last formula takes the form

$$\lambda_{h(k)} v^k = 0 . \tag{4.14}$$

Thus transvecting the formula (4.3) by v^k and using the above relation, we get

$$\lambda_h M_k v^k = 0, \text{ i.e. } M_k v^k = 0 . \tag{4.15}$$

Now, multiplying the relation (4.7) by v^m and taking care of the equations (1.15a) and the last formula, we obtain

$$E_{j(m)} = 0 . \tag{4.16}$$

Hence, when we have (1.15a), we can get an important property :

$$\psi_j E_h v^h = 0, \quad \text{i.e.} \quad E_h v^h = 0 . \tag{4.17}$$

Therefore, transvecting the formula (3.7) by v^h and noting the last equation, we obtain

$$\psi_{h(k)} v^h + \psi_h \psi_k v^h = 0 , \tag{4.18}$$

i.e.

$$(\psi_h v^h)_{(k)} = 0, \quad \text{i.e.} \quad \psi_h v^h = \text{const.}, \tag{4.19}$$

where we have used the latter part of (1.11). This completes the proof of $\psi_h v^h = \text{const.}$

Now, under (3.14b), i.e. (4.3) and (3.7) followed from (3.6) respectively, we have found the condition (4.10). And with the help of the equation (4.10), we can have

$$(\psi_{h(k)} - \psi_{k(h)})_{(m)} = \lambda_m (\psi_{h(k)} - \psi_{k(h)}) . \tag{4.20}$$

Namely in the present case (4.19) and (1.15a) we are able to say that the antisymmetric tensor $(\psi_{h(k)} - \psi_{k(h)})$ is not a zero-tensor, but a recurrent one with respect to the given vector λ_m in the space.

Consequently, in order to avoid getting $\psi_{h(k)} = \psi_{k(h)}$ appearing in (1.15b), that is, to get (1.15a) purely, it is sufficient to assume the non-parallel property of $(\psi_{h(k)} - \psi_{k(h)})$ instead of $H^i{}_{hjk} v^h \neq 0$ giving $(\psi_{h(k)} - \psi_{k(h)}) v^i \neq 0$, i.e. $\psi_{h(k)} \neq \psi_{k(h)}$.

Next, we have found in the above discussion that (2.6) and (4.10) followed from (3.6) and (3.6) may be proved conversely by use of a set of (2.6) and (4.10). Hence (3.6) and the set of (2.6) and (4.10) are equivalent with each other, furthermore, (3.6) gives us actually the desired projective recurrent affine motion.

By this reason, we can state :

Conclusion 4.1. In a non symmetric and non-flat PR Fn-space admitting projective affine motion, in order that we have a projective recurrent affine motion (1.11) having (1.12) and satisfying (1.15a), it is necessary and sufficient that we put, in the space, the conditions (2.6) and (4.10), where $(\psi_{h(k)} - \psi_{k(h)})$ should be assumed to be a non-parallel tensor in the space. Namely, comparing these conclusions for (1.15a), we can see that the condition (3.6) is more fruitful than the condition (2.6), so our pursued condition is (3.6) with an additional condition being $(\psi_{h(k)} - \psi_{k(h)})$ a non-parallel tensor.

Remark.

The author expresses his sincere thanks to Prof. K.Takano for the help taken from his papers in writing the series of papers on projective affine motion in Finsler space.

REFERENCES

[1] RUND, H. : The differential geometry of Finsler spaces, Springer-Verlag (1959).

[2] YANO, K. : The theory of Lie-derivatives and its applications, Amsterdam (1957).

[3] TAKANO, K. : *On a remained affine motion in Y.C. Wong's space*, Tensor, N.S., **13** (1963), 155-162.

[4] KUMAR, A. : *Projective affine motion in a PR Fn-space* (comm.).

[5] KUMAR, A. : *Projective affine motion in a PR Fn-space*, II (comm.).

[6] KUMAR, A. : *Projective affine motion in a PR Fn-space*, III (comm.).

[7] KUMAR, A. : *Projective affine motion in a PR Fn-space*, IV (comm.).

[8] KUMAR, A. : *Projective affine motion in a PR Fn-space*, IV (comm.).

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ÖZET

Bu çalışmada ψ_h nın gradiyent vektöründen farklı olduğu $\lambda_s v^s = 0$ hali incelenmekte ve böylece yazarın PR Fn uzayındaki projektif afin hareketlere dair, evvelce elde etmiş olduğu sonuçlar tamamlanmaktadır.