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SOME THEOREMS ON PROJECTIVE RECURRENCE VECTOR IN FINSLER SPACE

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In this paper it has been established that in a PR Fn-space the tensor $\lambda_{h(s)}$ is symmetric when projective Ricci tensor $H_{hj}(x, \dot{x})$ is symmetric iff there exists a contravariant vector $v^i(x)$ satisfying $\lambda_s v^s \equiv 1$, $H_{hj} v^h v^j = -\beta$ and $(\lambda_{h(s)} - \lambda_{s(h)}) v^a = 0$.

1. INTRODUCTION

Let us consider an n-dimensional affinelly connected Finsler space Fn [1][1)] having symmetric Berwald's connection coefficient G^i_{jk}. The covariant derivative of any tensor field T^i_j depending on both positional and directional coordinates is given by

$$T^i_{j(k)} = \partial_k T^i_j - \dot{\partial}_m T^i_j G^m_{\nu k} \dot{x}^\nu + T^h_j G^i_{hk} - T^i_h G^h_{jk}{}^{2)}, \qquad (1.1)$$

where

$$G^i_{\nu k} \dot{x}^\nu = G^i_k.$$

The well known commutation formula involving the above covariant derivative is given by

$$2T^i_{j[(h)(k)]} = -\dot{\partial}_\nu T^i_j H^\nu_{shk} \dot{x}^s + T^s_j H^i_{shk} - T^i_s H^s_{jhk}{}^{3)}, \qquad (1.2)$$

where

$$H^i_{hjk}(x, \dot{x}) \overset{\text{def.}}{=} 2\{\partial_{[j} G^i_{k]h} - G^i_{\nu h[j} G^\nu_{k]} + G^\nu_{h[j} G^i_{k]\nu}\} \qquad (1.3)$$

is called Berwald's curvature tensor and satisfies the following relations [1]:

$$H^i_{hk} = H^i_{jhk} \dot{x}^j, \qquad (1.4)$$

$$H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0, \qquad (1.5)$$

$$H^i_{hjk} = -H^i_{hkj}, \qquad (1.6)$$

$$\text{a) } H^i_{hji} = H_{hj} \quad \text{and} \quad \text{b) } H^i_{ihj} = H_{hj} - H_{jh}. \qquad (1.7)$$

In a Finsler space, if the Berwald's curvature tensor satisfies the relation

1) Numbers in square brackets refer to the references given at the end of the paper.

2) $\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$ and $\partial_i \equiv \partial/\partial x^i$.

3) $2 A_{[hk]} = A_{hk} - A_{kh}$.

$$H^i{}_{hjk(s)} = \lambda_s H^i{}_{hjk}, \tag{1.8}$$

where λ_s is any vector field, then the space is called projective recurrent Finsler space or PR Fn-space and λ_s is called projective recurrence vector.

2. DECOMPOSITION OF $H^i{}_{hjk}(x, \dot{x})$

In an affinelly connected Finsler space, the second Bianchi's identity for Berwald's curvature tensor $H^i{}_{hjk}(x, \dot{x})$ is given by

$$H^i{}_{hjk(s)} + H^i{}_{hks(j)} + H^i{}_{hsj(k)} = 0. \tag{2.1}$$

By virtue of the basic condition (1.8), the last identity can be written as

$$\lambda_s H^i{}_{hjk} + \lambda_j H^i{}_{hks} + \lambda_k H^i{}_{hsj} = 0. \tag{2.2}$$

On the other hand, transvecting the last formula by v^s and summing over the index s, we find

$$H^i{}_{hjk} = \lambda_k Q^i{}_{hj} - \lambda_j Q^i{}_{hk}, \tag{2.3}$$

where

$$Q^i{}_{hj} \overset{\text{def.}}{=\!=} H^i{}_{hjs} v^s \tag{2.4}$$

and

$$\lambda_s v^s = 1. \tag{2.5}$$

In view of the equations (1.6), (1.7b) and (2.5), we can get

$$\text{a) } Q^i{}_{hj} v^j = 0 \quad \text{and} \quad \text{b) } Q^i{}_{ij} = 0. \tag{2.6}$$

Now, if $Q^{*i}{}_{hj}$ is any tensor satisfying

$$H^i{}_{hjk} = \lambda_k Q^{*i}{}_{hj} - \lambda_j Q^{*i}{}_{hk}. \tag{2.7}$$

Then, by substraction (2.7) from (2.3), we can get

$$\lambda_k (Q^{*i}{}_{hj} - Q^i{}_{hj}) = \lambda_j (Q^{*i}{}_{hk} - Q^i{}_{hk}). \tag{2.8}$$

In view of the last formula, we can introduce a tensor $E_j{}^i$:

$$Q^{*i}{}_{hj} = Q^i{}_{hj} + \lambda_j E_h{}^i. \tag{2.9}$$

Conversely, if $Q^i{}_{hj}$ satisfies (2.3) and $E_h{}^i$ is any tensor, then the tensor $Q^{*i}{}_{hj}$ satisfies (2.7). Such a tensor $Q^{*i}{}_{hj}$ of the form (2.9) may be regarded as a symmetric one. We shall prove this fact. For this purpose introducing the formula (2.3) into the left-hand side of the identity (1.5), we obtain

$$\lambda_k (Q^i{}_{h} - Q^i{}_{jh}) + \lambda_h (Q^i{}_{jk} - Q^i{}_{kj}) + \lambda_j (Q^i{}_{kh} - Q^i{}_{hk}) = 0. \tag{2.10}$$

Next, transvecting the above formula by v^k and taking notice of the equation (2.5), we have

$$Q^i_{hj} - Q^i_{jh} = \lambda_h E_j^i - \lambda_j E_h^i \tag{2.11}$$

or

$$Q^i_{hj} + \lambda_j E_h^i = Q^i_{jh} + \lambda_h E_j^i , \tag{2.12}$$

where we have put

$$E_j^i \overset{\text{def.}}{=} (Q^i_{hj} - Q^i_{jh})\, v^h . \tag{2.13}$$

With the help of the equations (2.4) and (2.6) the above formula reduces to

$$E_j^i = Q^i_{hj}\, v^h = H^i_{hjk}\, v^k\, v^h . \tag{2.14}$$

In view of the equations (2.9) and (2.11), we can deduce

$$Q^{*i}_{hj} = Q^{*i}_{jh} . \tag{2.15}$$

This completes the proof.

3. DISCUSSION

Equating the formula (2.7) with respect to the indices i and k, we find

$$H_{hj} = \lambda_i\, Q^{*i}_{hj} - \lambda_j\, Q^{*i}_{hk} , \tag{3.1}$$

where we have used (1.7a).

Let us assume the symmetry of H_{hj}, then the above result yields

$$\lambda_j\, Q^{*i}_{hi} = \lambda_h\, Q^{*i}_{ji} , \tag{3.2}$$

where we have used (2.15).

The above relation gives us

$$Q^{*i}_{hi} = \beta\, \lambda_h . \tag{3.3}$$

By virtue of the equations (2.6b), (2.9) and (2.15), we can construct

$$Q^{*i}_{hi} = Q^{*i}_{ih} = Q^i_{ih} + \lambda_h E_i^i = \lambda_h E_i^i . \tag{3.4}$$

Comparing this equation with (3.3), we find

$$\beta = E_i^i . \tag{3.5}$$

In view of the equations (1.6), (1.7a), (2.4) and (2.14), the above relation can be also re-written as

$$\beta = E_i^i = Q^i_{hi}\, v^h = (-H_{hs}\, v^s\, v^h) , \tag{3.6}$$

i.e.

$$H_{hs}\, v^h\, v^s = -\beta . \tag{3.7}$$

By virtue of the basic condition (1.8) and the formula (2.14), we can get

$$\lambda_j\, E_h^i = \lambda_j\, H^i_{khs}\, v^s\, v^k = (H^i_{hhs(j)})\, v^s\, v^k . \tag{3.8}$$

From the equations (1.6) and (3.8), we can obtain

$$\lambda_j E_h^i - \lambda_h E_j^i = (H^i{}_{khs(j)} + H^i{}_{ksj(h)})\, v^s v^k . \tag{3.9}$$

In view of the identity (2.1), the above formula yields

$$\lambda_j E_h^i - \lambda_h E_j^i = - H^i{}_{kjh(s)}\, v^s v^k = \lambda_s v^s H^i{}_{khj} v^k = H^i{}_{khj} v^k , \tag{3.10}$$

where we have used (1.6) and (2.5).

Now, let us consider the defining equation (1.8) as a differential equation on $H^i{}_{hjk}(x, \dot{x})$. Then its integrability condition takes the form

$$\begin{aligned} M_{ms} H^i{}_{hjk} = &- \dot{\partial}_a H^i{}_{hjk} H^a{}_{nsm} \dot{x}^n + H^a{}_{hjk} H^i{}_{asm} - H^i{}_{ajk} H^a{}_{hsm} - \\ &- H^i{}_{hak} H^a{}_{jsm} - H^i{}_{hja} H^a{}_{ksm} , \end{aligned} \tag{3.11}$$

where we have put

$$M_{ms} \stackrel{\text{def.}}{=} (\lambda_{s(m)} - \lambda_{m(s)}). \tag{3.12}$$

Contracting the formula (3.11) with respect to the indices i and k and using the equation (1.7a), we obtain

$$M_{ms} H_{hj} = - \dot{\partial}_a H^i{}_{hji} H^a{}_{nsm} \dot{x}^n - H_{aj} H^a{}_{hsm} - H_{ha} H^a{}_{jsm} . \tag{3.13}$$

Transvecting the last formula by $v^h v^j$ and taking notice of (3.7), and the symmetry property of the projective Ricci tensor $H_{hj}(x, \dot{x})$, we have

$$\beta M_{ms} = 2 H^a{}_{hsm} v^h (H_{aj} v^j). \tag{3.14}$$

On the other hand, introducing the formula (3.10) into the right-hand side of the above result, we get

$$\beta M_{ms} = 2 (\lambda_m E_s^a - \lambda_s E_m^a)(H_{aj} v^j). \tag{3.15}$$

Transvecting the formula (2.13) by v^j and noting the relations (2.6a), we get

$$E_j^i v^j = 0. \tag{3.16}$$

In this way, multiplying the formula (3.15) by v^m and using the equations (2.5) and (3.16), we find

$$\beta M_{ms} v^m = 2 E_s^a H_{aj} v^j \tag{3.17}$$

or

$$\beta \lambda_h M_{ms} v^m = 2 E_s^a \lambda_h H_{aj} v^j . \tag{3.18}$$

On an analogous way, we can obtain

$$\beta \lambda_s M_{hm} v^m = - 2 \lambda_s E_h^a H_{aj} v^j . \tag{3.19}$$

Adding (3.18) and (3.19) side by side and remembering the formula (3.15), we find

$$\beta (\lambda_h M_{ms} v^m + \lambda_s M_{hm} v^m) = \beta M_{hs} , \tag{3.20}$$

i.e.

$$M_{hs} = \lambda_h M_{ms} v^m - \lambda_s M_{mh} v^m , \tag{3.21}$$

where we have used $M_{mh} = - M_{hm}$ and $\beta \neq 0$.

Consequently, when and only when, we have

$$\lambda_h M_{ms} v^m = \lambda_s M_{mh} v^m \tag{3.22}$$

we find

$$M_{hs} = 0, \text{ i.e. } \lambda_{s(h)} = \lambda_{h(s)} . \tag{3.23}$$

Multiplying the formula (3.22) by v^h and taking care of the equation (2.5), we have

$$M_{mh} v^m = 0 , \tag{3.24}$$

where we have also used

$$M_{mh} v^m v^h = 0 . \tag{3.25}$$

Conversely from the formula (3.24) we can get (3.22). Hence (3.22) and (3.24) are equivalent to each other. Thus we can state the following:

Conclusion. In a PR Fn-space, if projective Ricci tensor $H_{hj}(x, \dot{x})$ is symmetric when and only when there exists a contravariant vector $v^i(x)$ satisfying (2.5), $H_{hj} v^h v^j$ and $(\lambda_{h(s)} - \lambda_{s(h)}) v^s = 0$, the tensor $\lambda_{h(s)}$ is symmetric.

4. REMARKS

Let us consider the case of $\beta = 0$. According to (3.20), such a case occurs when we have $M_{hs} - (\lambda_h M_{ms} + \lambda_s M_{hm}) v^m \neq 0$.

Transvecting the equation (3.13) by v^m side by side and summing over m we get

$$M_{ms} v^m H_{hj} = - H_{aj} H^a{}_{hsm} v^m - H_{ha} H^a{}_{jsm} v^m. \tag{4.1}$$

Now, again transvecting the above formula by λ_k and using the basic condition (1.8), we get

$$\lambda_k M_{ms} v^m H_{hj} = - (H^a{}_{hsm(k)}) H_{aj} v^m - (H^a{}_{jsm(k)}) v^m H_{ha} . \tag{4.2}$$

On an analogous way, we can obtain

$$\lambda_s M_{km} v^m H_{hj} = (H^a{}_{hkm(s)}) H_{aj} v^m + (H^a{}_{jkm(s)}) v^m H_{ha} , \tag{4.3}$$

where we have also used the non-symmetric property of M_{km}.

Adding the reductions (4.2) and (4.3) side by side and noting the identity (2.1), we find

$$(\lambda_k M_{ms} - \lambda_s M_{mk}) v^m H_{hj} = - H^a{}_{hsk} H_{aj} - H^a{}_{jsk} H_{ha} , \tag{4.4}$$

where we have used (2.5) and $M_{ha} = - M_{ah}$.

In view of the equation (3.13), the above relation yields

$$(\lambda_k M_{ms} - \lambda_s M_{mk}) v^m H_{hj} = M_{ks} H_{hj}, \tag{4.5}$$

i.e.

$$[M_{ks} - (\lambda_k M_{ms} - \lambda_s M_{mk}) v^m] H_{hj} = 0, \tag{4.6}$$

from which by assumption we have $H_{hj} = 0$. This is absurd. Consequently such a case of $\beta = 0$ is not relevant to the subject.

5. ANOTHER CONCLUSION

In § 3, we have introduced a proportional factor β and we have obtained (3.7). And, by use of this factor β, we have devised to find (3.20), from which under $\beta \neq 0$, we have found the condition for the symmetry of $\lambda_{h(s)}$.

However, as we have mentioned in § 4, from (3.13), we have to take care of the fact that, having no connection with $H_{hj} = H_{jh}$, the formula (4.6) has been derived from (3.13). Furthermore (3.13) has not also any connection with the symmetry of H_{hj}, but it has been derived from the defining equation itself of the space. And, in this care, assuming $H_{hj} \neq 0$ we have (3.21). Thus, we can see, by use of (2.5), the condition for the gradience of λ_k as $M_{hk} v^h = 0$.

Considering these facts, we can state that the existence of non-vanishing β is related with the symmetry assumption of H_{hj}. According to these facts we are able to get here another conclusion having no connection with $H_{hj} = H_{jh}$.

Conclusion. In a PR Fn-space with a non-vanishing projective Ricci tensor, the recurrence vector λ_h is a gradient one, when and only when there exists a contravariant vector v^i satisfying (2.5) and $(\lambda_{h(s)} - \lambda_{s(h)}) v^s = 0$.

6. CONTINUED DISCUSSION

In order to get the second supposition, i.e. (3.7) in the conclusion of § 3 it is sufficient to take

$$H_{hj} v^j = -\beta \lambda_h \quad (\beta \neq 0) \tag{6.1}$$

into consideration, because contraction of the above formula by v^h and (2.5) gives us (3.7).

By virtue of the equations (1.7a), (3.6) and (3.10), we can construct

$$H_{jh} v^j = -\beta \lambda_h + \lambda_j E_h^j. \tag{6.2}$$

Next, comparing the equations (6.1) and (6.2), we get

$$\lambda_j E_h^j = 0. \tag{6.3}$$

Multiplying the formula (6.1) by E_s^h and using the last equation, we have

$$H_{hj}\, E_h^s\, v^j = 0. \tag{6.4}$$

Thus, with the help of the equations (3.17) and (6.4) we can deduce

$$\beta\, H_{ks}\, v^k = 0. \tag{6.5}$$

Since $\mu \neq 0$, the last formula can be rewritten as

$$H_{ks}\, v^k = 0. \tag{6.6}$$

This is the third supposition in the conclusion of § 3. Thus, the conclusion in § 3 may be replaced by the next.

CONCLUSION. In a PR Fn-space, when the projective Ricci tensor $H_{hj}(x, \dot{x})$ is symmetric, if we choose a contravariant vector v^i so as to satisfy (2.5) and (6.1), where β means a scalar function, then $\lambda_{h(s)}$ becomes a symmetric tensor.

It is easy to see that (6.1) is equivalent to (6.3). In fact, as we have (6.2) always, if (6.3) will be the case, we can get (6.1) and conversely. On the other hand transvecting the formula (3.10) by λ_i side by side and summing over the index i, we obtain

$$\lambda_i\, (\lambda_j\, E_h^i - \lambda_h\, E_j^i) = \lambda_i\, H^i{}_{khj}\, v^k\,. \tag{6.7}$$

By virtue of the formula (6.3), the above relation takes the form

$$\lambda_i\, H^i{}_{khj}\, v^k = 0 \tag{6.8}$$

or

$$(H^i{}_{khj(i)})\, v^k = 0. \tag{6.9}$$

Next, in view of the equations (1.7), (1.8) and (2.1), the last formula becomes

$$\lambda_h\, H_{kj}\, v^k = \lambda_j\, H_{kh}\, v^k\,. \tag{6.10}$$

Transvecting the last formula by v^h and taking care of the equations (2.5) and (3.7), we can find

$$H_{kj}\, v^k = -\, \beta \lambda_j\,. \tag{6.11}$$

Namely from (6.8) we can get (6.1), consequently (6.3). By this reason, (6.8) is equivalent to (6.3), i.e. (6.3) is able to be replaced by (6.8). Thus from the above conclusion we can have :

Theorem 6.1. In a PR Fn when the projective Ricci tensor is symmetric, in order to get $\lambda_{h(s)} = \lambda_{s(h)}$ it is sufficient that there exists a contravariant vector v^i satisfying (2.5) and (6.8).

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ÖZET

Bu çalışmada, bir PR Fn uzaymdaki $\lambda_{h(s)}$ tensörünün, $H_{hj}(x,\dot{x})$ projektif Ricci tensörünün simetrik olmasının $\lambda_s v^s \equiv 1$, $H_{hj} v^h v^j = -\beta$ ve $(\lambda_{h(s)} - \lambda_{s(h)}) v^s = 0$ koşullarıın sağlayan bir kontravaryant $v^i(x)$ vektörünün varlığına denk olması haliude simetrik olacağı saptanmaktadır.